On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

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Motivations

Kleinian singularities are the only normal Gorenstein singularities in dimension 2. Anti-Poisson involutions and their fixed point loci appear naturally when we want to classify irreducible Harish-Chandra modules over Kleinian singularities.

Overview:

minimal resolution		Kleinian singularity	
$\widetilde{X}$	$\xrightarrow{\pi}$	$X \circlearrowleft \theta$	anti-Poisson involution
$\cup$		$\cup$	
$\pi^{-1}(X^\theta)$		X^θ	Lagrangian subvariety

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as schemes.

1. Kleinian singularities

2. Anti-Poisson involutions and their fixed point loci

3. Preimage of fixed point loci under minimal resolutions

Kleinian singularities

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. The algebra of invariant functions $\mathbb{C}[u, v]^\Gamma$ is finitely generated.

Kleinian singularities are the quotients $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u, v]^\Gamma$.

Example

When $\Gamma = \{\pm I_2\}$, we have $\mathbb{C}[u, v]^\Gamma =$ even degree polynomials $=$ $\mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x, y, z]/(xy - z^2)$.

Fact (Klein, 1884)

$\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation), and $\mathbb{C}^2/\Gamma$ has an isolated singularity at 0.

Kleinian singularities

Classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$**:**

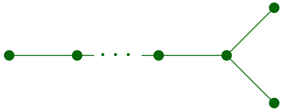
- The cyclic group of order $n+1$. A_n
  $xy - z^{n+1} = 0$
- The binary dihedral group of order $4(n-2)$, $n \geq 4$. D_n
  $x^{n-1} + xy^2 + z^2 = 0$
- The binary tetrahedral group of order 24. E_6
  $x^4 + y^3 + z^2 = 0$
- The binary octahedral group of order 48. E_7
  $x^3y + y^3 + z^2 = 0$
- The binary icosahedral group of order 120. E_8
  $x^5 + y^3 + z^2 = 0$

McKay correspondence: Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are in bijection with ADE Dynkin diagrams.

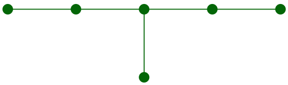
Kleinian singularities

A_n

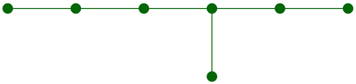
D_n



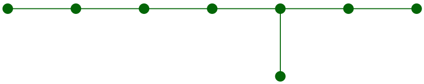
E_6



E_7



E_8



McKay correspondence

How to attach a Dynkin diagram to a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$?

1. McKay Correspondence [McKay, 1979].
2. Minimal Resolution [Du Val, 1934]. By a resolution one means a smooth variety $\widetilde{X}$ equipped with a projective birational morphism $\pi\colon \widetilde{X} \to X := \mathbb{C}^2/\Gamma$. The minimality condition means that any other resolution factors through $\widetilde{X}$. The exceptional fiber

$$\pi^{-1}(0) = C_1 \cup \cdots \cup C_n,\ \ C_i \simeq \mathbb{P}^1$$

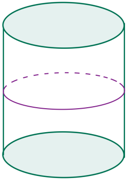
is a connected union of $\mathbb{P}^1$'s. We can construct the dual graph of $\pi^{-1}(0)$ by replacing each C_i by a vertex i and joining vertices i and j by an edge if C_i intersects with C_j.

Fact (Du Val, 1934)

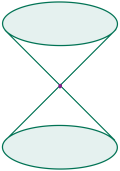
The dual graph of $\pi^{-1}(0)$ is the corresponding type Dynkin diagram.

Example: A_1 and D_4 singularities

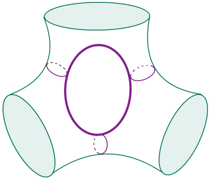
$T^*\mathbb{P}^1$



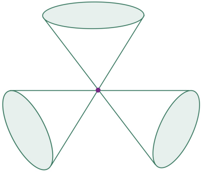
$\xrightarrow{\pi}$



A_1 singularity
$xy - z^2 = 0$



$\xrightarrow{\pi}$



D_4 singularity
$x^3 + xy^2 + z^2 = 0$

Anti-Poisson involutions

Set $X := \mathbb{C}^2/\Gamma$. The algebra of functions $\mathbb{C}[X] = \mathbb{C}[u, v]^\Gamma$ is a graded (*by degree of polynomials in* u, v) Poisson algebra with Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

Example

On type $A_n : \mathbb{C}[x, y, z]/(xy - z^{n+1})$ Kleinian singularity. The Poisson brackets are given by

$$\{x, y\} = (n+1)^2 z^n,$$
$$\{x, z\} = (n+1)x,$$
$$\{y, z\} = -(n+1)y.$$

Anti-Poisson involutions

Definition

An *anti-Poisson involution* of a Kleinian singularity $X := \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\},\ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

Example

On type A_n: $\mathbb{C}[x, y, z]/(xy - z^{n+1})$ Kleinian singularity,

$$x \mapsto y,\ \ y \mapsto x,\ \ z \mapsto z$$

is an anti-Poisson involution.

Anti-Poisson involutions

Proposition 1 (H.)

There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by graded Poisson automorphisms.

Anti-Poisson involutions for Kleinian singularities of type A_n:

	case I	case II	case III
A_n, n odd $xy - z^{n+1} = 0$	$x \mapsto y$ $y \mapsto x$ $z \mapsto z$	$x \mapsto x$ $y \mapsto y$ $z \mapsto -z$	$x \mapsto -x$ $y \mapsto -y$ $z \mapsto -z$
A_n, n even $xy - z^{n+1} = 0$	$x \mapsto y$ $y \mapsto x$ $z \mapsto z$	$x \mapsto -x$ $y \mapsto y$ $z \mapsto -z$	

D_n, two cases; E_6, two cases; E_7 one case; E_8, one case.

The fixed point loci

$X = \mathbb{C}^2/\Gamma$. Define the *fixed point locus* $X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$.

Example (continued)

Type A_n Kleinian singularity $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1})$ with θ swapping $x \leftrightarrow y$. We have

$$X^\theta = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1}, x - y) \simeq \operatorname{Spec} \mathbb{C}[x, z]/(x^2 - z^{n+1}),$$

which is a union of two $\mathbb{A}^1$'s when n is odd, a cusp when n is even.

Proposition 2 (H.)

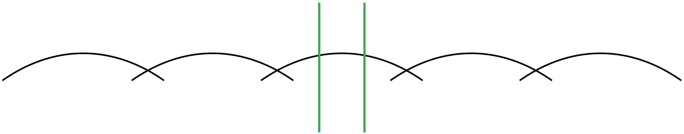
- *The fixed point locus X^θ is reduced.*
- *If X^θ is not a single point, each irreducible component of X^θ is either $\mathbb{A}^1$ or a cusp.*

Preimage of fixed point loci under minimal resolutions

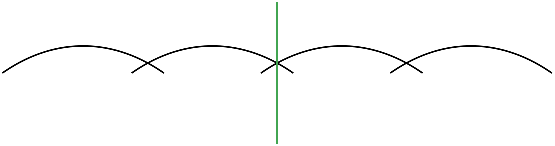
Recall $\pi\colon \widetilde{X} \to X = \mathbb{C}^2/\Gamma$ denotes the minimal resolution. We would like to describe the preimage $\pi^{-1}(X^\theta)$. $0 \in X^\theta \Rightarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$.

Example: Consider type A_n singularities with θ swapping $x \leftrightarrow y$.

X^θ $\pi^{-1}(X^\theta)$



A_n, n odd $n = 5$



A_n, n even $n = 4$

Straight green line is $\mathbb{A}^1$, curly black line is $\mathbb{P}^1$.

Preimage of fixed point loci under minimal resolutions

$\pi\colon \widetilde{X} \to X = \mathbb{C}^2/\Gamma$, minimal resolution. Want to describe $\pi^{-1}(X^\theta)$.

Q1: What are the irreducible components?

Q2: How do different components intersect with each other?

Q3: Is $\pi^{-1}(X^\theta)$ reduced or not?

To answer these questions, we need to lift anti-Poisson involutions of X to their minimal resolutions $\widetilde{X}$. By a *lift* of θ, we mean an anti-symplectic involution $\tilde{\theta}\colon \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \tilde{\theta} = \theta \circ \pi$.

$$\pi^{-1}(X^\theta) = \pi^{-1}(0) \cup \widetilde{X}^{\tilde{\theta}}.$$

Fact: $\widetilde{X}$ smooth $\Rightarrow \widetilde{X}^{\tilde{\theta}}$ is smooth Lagrangian.

Theorem 3 (H.)

There exists a unique anti-symplectic involution $\tilde{\theta}$ of $\widetilde{X}$ that lifts θ.

Preimage of fixed point loci under minimal resolutions

Q1: What are the irreducible components?

$\mathbb{A}^1$'s and $\mathbb{P}^1$'s, where $\mathbb{A}^1$'s are normalizations of the irreducible components of X^θ, and $\mathbb{P}^1$'s are irreducible components of $\pi^{-1}(0)$.

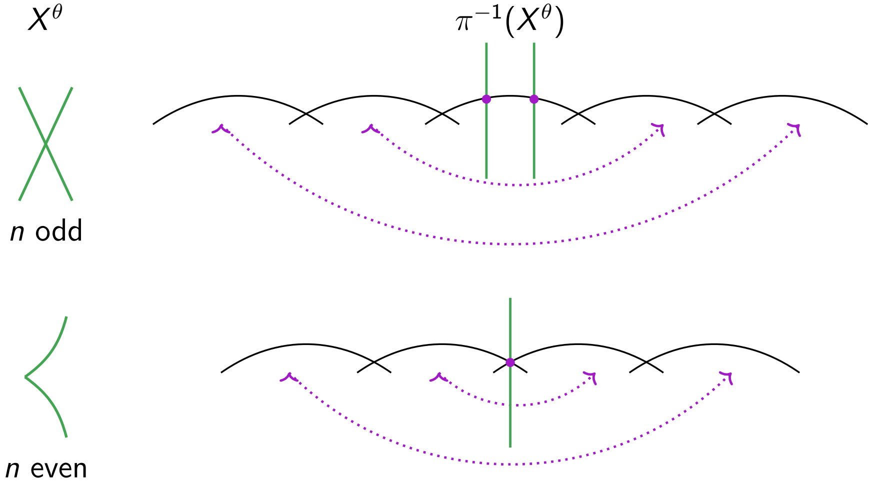
Q2: How do different components intersect with each other?

- $\mathbb{A}^1$'s do not intersect with each other (as they are contained in $\widetilde{X}^{\tilde{\theta}}$).
- $\mathbb{P}^1$'s intersect with each other according dually to a Dynkin diagram.
- $\mathbb{A}^1$ intersects with a unique $\mathbb{P}^1$ at an isolated $\tilde{\theta}$-fixed point of $\pi^{-1}(0)$.

Preimage of fixed point loci under minimal resolutions

Example: Consider type A_n singularities with θ swapping $x \leftrightarrow y$.

The action of $\tilde{\theta}$ on $\pi^{-1}(0)$ is given by the dotted purple arrows.



Preimage of fixed point loci under minimal resolutions

Q3: Is $\pi^{-1}(X^\theta)$ reduced?

No in general. Write $\pi^{-1}(X^\theta) = \sum_{j=1}^m 1 \cdot L_j + \sum_{i=1}^n a_i C_i$ as a divisor, where $L_j \simeq \mathbb{A}^1$, $C_i \simeq \mathbb{P}^1$.

Set $b_i :=$ # of $\mathbb{A}^1$'s that a $\mathbb{P}^1$ intersects with.

Proposition 4 (H.)

If $X^\theta \subset X$ is a principal divisor, then

$$(a_1, \cdots, a_n)^t = \mathcal{C}^{-1}(b_1, \cdots, b_n)^t,$$

where $\mathcal{C}$ is the corresponding Cartan matrix of types ADE.

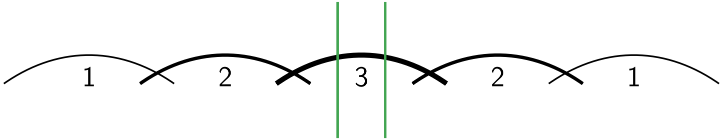
Remark: In almost all cases, X^θ is a principal divisor on X.

Preimage of fixed point loci under minimal resolutions

Example (continuted): Type A_5 singularity with θ swapping $x \leftrightarrow y$.

X^θ

$\pi^{-1}(X^\theta)$



$$(b_1, b_2, b_3, b_4, b_5) = (0, 0, 2, 0, 0)$$
$$(a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 2, 1)$$

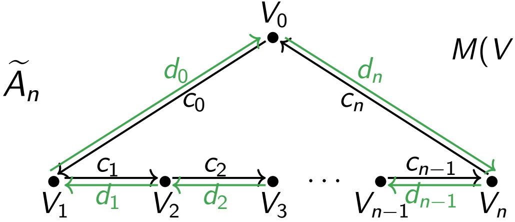
Proof of Main Theorem

Theorem 3 (H.)

There exists a unique anti-symplectic involution $\tilde{\theta}$ of $\widetilde{X}$ that lifts θ.

Idea of proof: realize Kleinian singularity as Nakajima quiver variety.

Example: Kleinian singularity of type A_n: $xy - z^{n+1}$ as a quiver variety.



$$\dim V_i = 1$$

$$M(V) := T^*(\bigoplus_{i=0}^n \mathrm{Hom}(V_i, V_{i+1}))$$

$$G := \prod_{i=0}^n \mathrm{GL}(V_i) \curvearrowright M(V)$$

$$\mu : M(V) \to \oplus_{i=0}^n \mathfrak{gl}(V_i)$$

$$\mu_i = c_{i-1}d_{i-1} - d_i c_i$$

We have $\mu^{-1}(0)/\!/G \simeq \mathrm{Spec}\, \mathbb{C}[x, y, z]/(xy - z^{n+1})$ with

$$x := c_n c_{n-1} \cdots c_1 c_0, \ \ y := d_0 d_1 \cdots d_{n-1} d_n, \ \ z := c_0 d_0.$$

The lift of θ: $x \leftrightarrow y$ is $\tilde{\theta}$: $c_i \leftrightarrow d_{n-i}$.

Thank you for your attention!