UCLA, Mar 10,

On certain Lagrangian subvarieties in minimal resolution of Kleinian singularities

overview + motivation

1) Kleinian singularities
2) Anti-Poisson involution & their fixed loci
   Main Thm
3) Preimage of fixed loci

extra time: talk about proof of Main Thm.

Overview:

minimal resolution   Kleinian singularity (throughout)

$$\begin{array}{ccccl} \tilde{X} & \xrightarrow{\pi} & X & \circlearrowleft \ \theta & \text{anti-Poisson involution} \\ \cup & & \cup & & \\ \pi^{-1}(X^\theta) & & X^\theta & & \text{(singular) Lagrangian subvariety.} \end{array}$$

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as schemes.
- irreducible components
- intersection pattern
- reduced or not (multiplicity)

<u>Motivation</u>: Classification of irreducible $HC(\mathfrak{g}, K)$-mod

- G. simple simply-connected algebraic group $/\mathbb{C}$; $\mathfrak{g}$. $\mathcal{N}$

  $\sigma: \mathfrak{g} \longrightarrow \mathfrak{g}$. Lie algebra involution $\rightsquigarrow$ $\mathfrak{g} = \underbrace{\overset{+1}{\mathfrak{k}} \oplus \overset{-1}{\mathfrak{p}}}$. $\longrightarrow$ K, corresponding connected alg group

  $\theta := -\sigma$ anti-Poisson involution on $\mathfrak{g}$

- $\mathbb{O}$, nilpotent orbit. $\mathbb{O}' \subset \overline{\mathbb{O}}$, $\operatorname{codim}_{\overline{\mathbb{O}}} \mathbb{O}' = 2$

  $e' \in \mathbb{O}'$, a normal point in $\overline{\mathbb{O}}$ (o.w. take normalization)

  $\downarrow$

  S'. Slodowy slice $\rightsquigarrow$ $S' \cap \overline{\mathbb{O}} \simeq X$. a Kleinian singularity

  θ restricts to $S' \cap \overline{\mathbb{O}}$ $\rightsquigarrow$ $(S' \cap \overline{\mathbb{O}})^{\theta} \simeq X^{\theta}$, fixed locus

  $(S' \cap \overline{\mathbb{O}})^{\theta} = S' \cap \overline{\mathbb{O}} \cap \mathfrak{p}$

- <u>A $HC(\mathfrak{g}, K)$-module</u> is a f.g. $U(\mathfrak{g})$-mod, M, s.t $\mathfrak{k} \curvearrowright M$ locally finitely & integrates to $K \curvearrowright M$.

  $U(\mathfrak{g})$, PBW filtration

  M, good filtration: K-stable, compatible with $U(\mathfrak{g})$

Associated variety $AV(M) := \operatorname{Supp}(\operatorname{gr} M)$ (set-theoretic) $\subset \mathfrak{p}$ (b/c K-stable filtration)

M irreducible $\rightsquigarrow$ $AV(M) \subset \mathcal{N} \cap \mathfrak{p}$

$\mathbb{O}$ nilpotent orbit $\rightsquigarrow$ $J(\mathbb{O})$ unipotent ideal associated with $\mathbb{O}$

$J(\mathbb{O}) := \operatorname{Ker}(U(\mathfrak{g}) \longrightarrow \mathcal{A}_0]$ where $\mathcal{A}_0$ is the canonical quantization of $\mathbb{C}[\mathbb{O}]$ (corresponds to parameter $0 \in H^2(\mathbb{O}, \mathbb{C})$)

Consider irreducible M annihilated by $J(\mathbb{O})$

- $\operatorname{codim}_{\overline{\mathbb{O}}} \partial\mathbb{O} \geq 4$, $AV(M) =$ closure of <u>single K-orbit</u> (dense by $\mathbb{O}_K$) in $\mathbb{O} \cap \mathfrak{p}$ [Vogan. 91]

  [Losev & Yu, 23] classified irreducible M annihilated by $J(\mathbb{O})$ with $AV(M) = \overline{\mathbb{O}_K}$

  $\longleftrightarrow$ twisted (by half-canonical twist) local system on $\mathbb{O}_K$

- $\operatorname{codim}_{\overline{\mathbb{O}}} \partial\mathbb{O} = 2$. $AV(M) \subset \overline{\mathbb{O}} \cap \mathfrak{p}$. but may not be irreducible.

  classification unknown! $\overset{\cap \text{ slice}}{\rightsquigarrow}$ $S' \cap \overline{\mathbb{O}} \cap \mathfrak{p} \simeq X^{\theta}$

  codim ≥ 4 $\rightsquigarrow$ non-isomorphic M $\rightsquigarrow$ non-isomorphic twisted local system

§ 1. Reminder on Kleinian singularities

Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup.

Kleinian singularity $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u, v]^\Gamma$

Example: $\Gamma = \{\pm I_2\}$ (A_1)

$\mathbb{C}[u, v]^\Gamma =$ even degree polynomials $= \mathbb{C}[x = u^2, y = v^2, z = uv]$

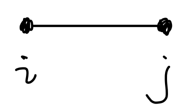
$= \mathbb{C}[x, y, z] / (xy - z^2)$

Theorem (Klein, 1884): $X = \mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (single relation) with an isolated singularity at 0.

minimal resolution $\pi : \underset{\text{smooth}}{\tilde{X}} \longrightarrow X$, projective & birational.

exceptional locus $\pi^{-1}(0)_{red} = \underbrace{\boxed{C_1}}_{\text{irreducible components}} \cup \cdots \cup C_n$. $\quad C_i \simeq \mathbb{P}^1$.

Dual graph of $\pi^{-1}(0)_{red}$: $\quad C_i \rightsquigarrow$ 

$C_i \cap C_j \neq \phi \rightsquigarrow$ 

Fact (Du Val, 1934): Dual graphs of $\pi^{-1}(0)_{red} \xleftrightarrow{\text{bijection}}$ ADE Dynkin diagrams
(McKay correspondence)

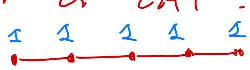
Rem: $\pi^{-1}(0)$ is not reduced in general!

$\mathfrak{g}$: simple Lie algebra of types ADE, simple root system $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$

δ - unique maximal root. $\delta = \sum_{i=1}^n \delta_i \alpha_i$. in the adjoint rep $\mathfrak{g} \curvearrowright \mathfrak{g}$.

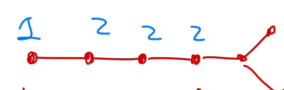
Then we have $\pi^{-1}(0) = \sum_{i=1}^n m_i C_i$ as a divisor. [Artin 66].

Type A_n: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\delta = \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \cdots + \alpha_n \rightsquigarrow \pi^{-1}(0)$ reduced



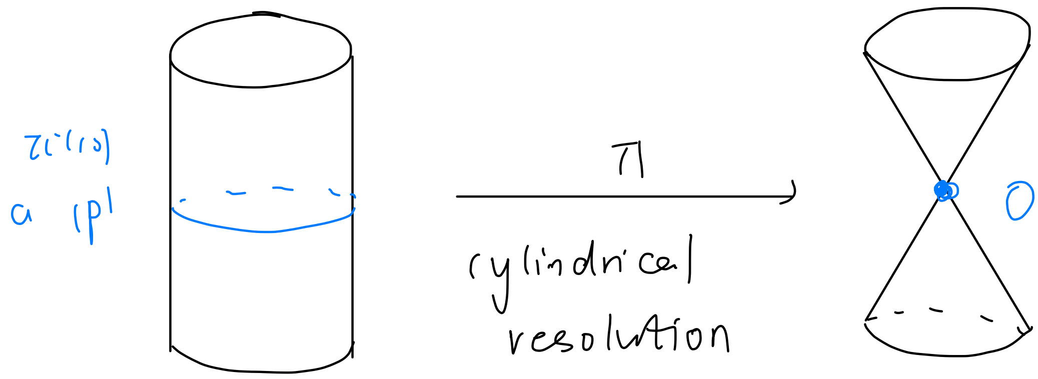
$(m_1 \cdots m_n) = (1, \cdots 1)$

Type D_n $\quad (\delta_1 \cdots \delta_n) = (1, 2, \cdots 2, 1, 1) \rightsquigarrow \pi^{-1}(0)$ not reduced



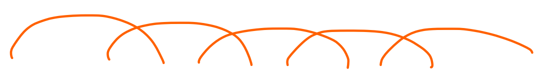
Examples:

1) A_1

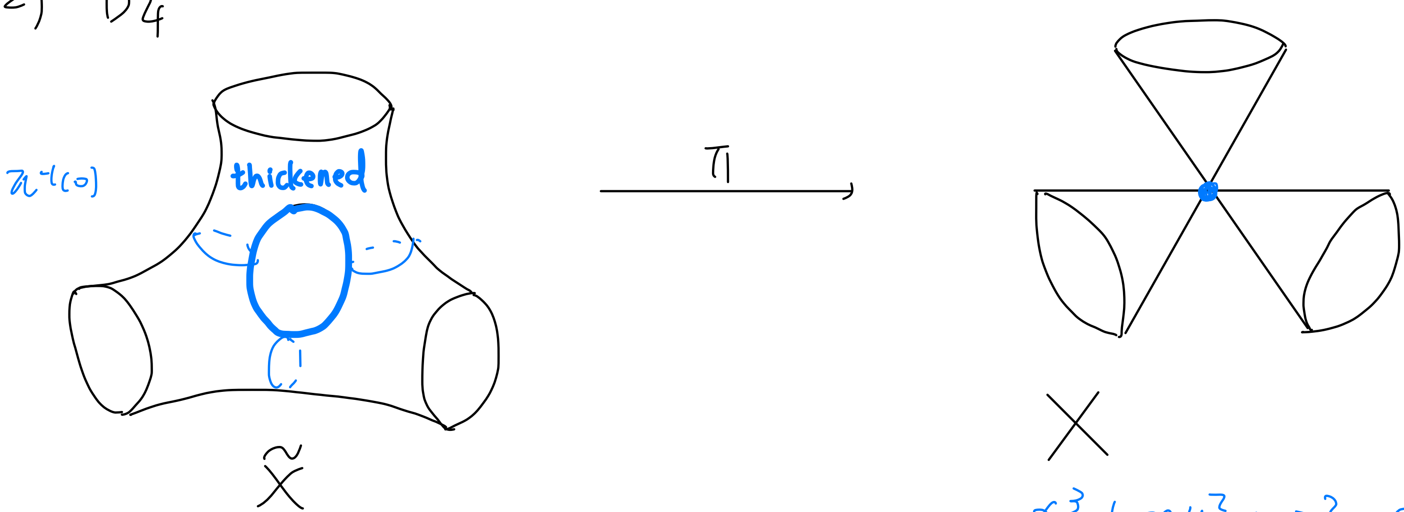


$\tilde{X} \simeq T^*\mathbb{P}^1$ $\qquad$ X $\quad xy - z^2 = 0$

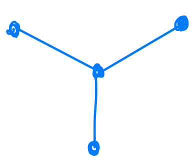
$\pi^{-1}(0)_{red} = \mathbb{P}^1$ $\quad$ dual graph  A_1 Dynkin diagrams

More generally, type A: $\pi^{-1}(0)$  (reduced)

2) D_4



$x^3 + xy^2 + z^2 = 0$

dual graph  D_4 Dynkin diagram

§2. Anti-Poisson involutions

$\mathbb{C}[u,v]$, graded by degree of polynomials

Poisson bracket $\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}$ (deg -2)

$\Gamma \subset SL_2(\mathbb{C}) \rightarrow \mathbb{C}[u,v]^\Gamma \subset \mathbb{C}[u,v]$,

graded Poisson subalgebra (deg -2)

Example: Type A_n: $\mathbb{C}[X] = \mathbb{C}[x,y,z]/(xy - z^{n+1})$

$$\{x, y\} = (n+1)^2 z^n$$

$$\{x, z\} = (n+1)x$$

$$\{y, z\} = -(n+1)y$$

Def: An <u>anti-Poisson involution</u> of a Kleinian singularity $X \simeq \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta: \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\} \quad \forall f_1, f_2 \in \mathbb{C}[X].$$

Example: Type A_n: $\theta: \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

$$x \longmapsto y, \quad y \longmapsto x, \quad z \longmapsto z$$

is an anti-Poisson involution.

Def (scheme-theoretic fixed locus):

$$X^\theta := \operatorname{Spec} \mathbb{C}[X]/I, \quad \text{where} \quad I = (\theta(f) - f \mid f \in \mathbb{C}[X])$$

Example (continued)

$$X^\theta = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^{n+1}, x-y) \simeq \operatorname{Spec} \mathbb{C}[x,z]/(x^2 - z^{n+1}) \text{ reduced.}$$

- union of two $\mathbb{A}^1$ when n odd
- a cusp when n even.

Prop 1 (Classification of θ)

conj. classes in $N_{GL_2}(\Gamma)/N_{SL_2}(\Gamma) \rightsquigarrow$ finite

There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$, up to conjugation by Poisson automorphisms.

Rem: They can be written out explicitly in terms of generators of X.

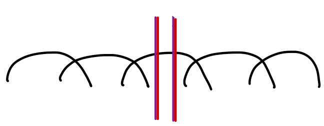
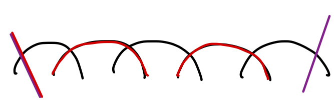
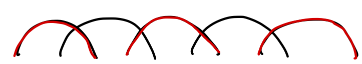
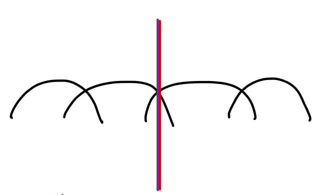
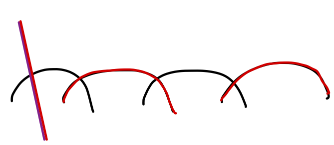
Prop 2 (Description of X^θ)

The scheme-theoretic fixed locus X^θ is reduced.

When $X^\theta \neq \{0\}$, each irreducible component is either an $\mathbb{A}^1$ or a cusp.

Example

Anti-Poisson involutions for Type A_n Kleinian singularity

Corresponding Lie alg involution (reg & subreg orbits)	Type I outer involution $\sigma: X \longmapsto -X^t$	Type II inner involutions $\sigma = Ad(I_{k,l})$	Type III not seen (excluding A_1)
A_n. n odd. $xy - z^{n+1} = 0$ X^θ	$x \longleftrightarrow y$ 	$z \longmapsto -z$ 	$x \longmapsto -x$ $y \longmapsto -y$ $z \longmapsto -z$ 
$\pi^{-1}(X^\theta)$, $\tilde{X}^{\tilde{\theta}}$			
A_n. n even $xy - z^{n+1} = 0$ X^θ	$x \longleftrightarrow y$ 	$x \longmapsto -x$ $z \longmapsto -z$ 	N/A
$\pi^{-1}(X^\theta)$, $\tilde{X}^{\tilde{\theta}}$ Cartan.	 the only triple intersection		

D_n, E_6. two types of API ; E_7, E_8 one type

§3 Preimage of fixed loci under minimal resolution

$\pi: \tilde{X} \longrightarrow X$, $0 \in X^{\theta} \rightsquigarrow \pi^{-1}(0) \subset \pi^{-1}(X^{\theta})$, but there are more; draw the picture

§3.1 lift.

Def: A lift of $\theta: X \longrightarrow X$ is an anti-symplectic involution $\tilde{\theta}: \tilde{X} \longrightarrow \tilde{X}$

s.t. $\pi \circ \tilde{\theta} = \theta \circ \pi$

Thm 3: There exists a unique lift for any anti-Poisson involution θ on $\mathbb{C}^2/\Gamma$

Claim: $\pi^{-1}(X^{\theta})_{red} = \underbrace{\pi^{-1}(0)_{red}}_{\text{union of } \mathbb{P}^1\text{'s}} \cup \boxed{\tilde{X}^{\tilde{\theta}}}$ $\quad \tilde{X}^{\tilde{\theta}}_{red}$ is smooth Lagrangian

no intersection! $\quad$ X becomes || in $\pi^{-1}(X^{\theta})$;

no cusp! $\quad$ { becomes | in $\pi^{-1}(X^{\theta})$

Lemma 1 (Exercise): (M, ω) symplectic manifold with anti-symplectic involution τ ($\tau^*\omega = -\omega$)
↳ pf: look at tangent space
Then M^{τ} is either empty or a Lagrangian submanifold.

Let $L \subset X^{\theta}$ be an irreducible component

define $\tilde{L} := \overline{\pi^{-1}(L \setminus 0)}$, then $\tilde{L}$ is an irreducible component in $\pi^{-1}(X^{\theta})$ &

$\tilde{L} \subset \tilde{X}^{\tilde{\theta}} \Rightarrow \tilde{L}$ smooth.

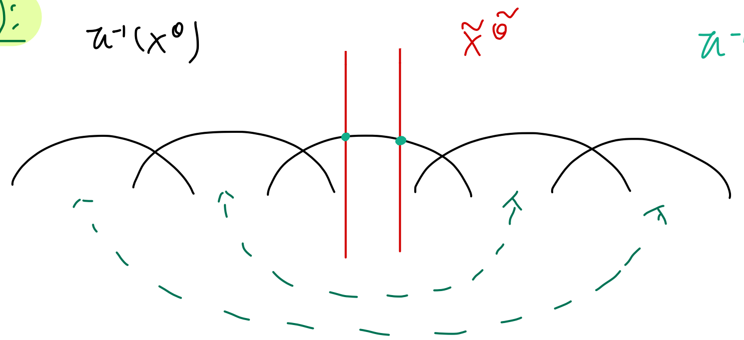
Claim $\tilde{L} \simeq \mathbb{A}^1 \rightsquigarrow \tilde{L} \cap \pi^{-1}(0)$ is a single (discrete) point in $\pi^{-1}(0)^{\tilde{\theta}}$.

Lemma 2 (Exercise): An involution on $\mathbb{P}^1$ either acts trivially or has exactly two fixed (discrete) points. (comes from Möbius transformation)

Example (continued):

Type A: $\theta: X \hookrightarrow Y$

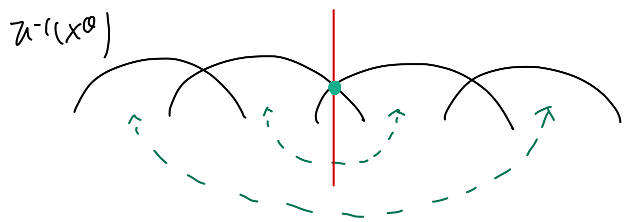
n odd



$\pi^{-1}(0)^{\tilde{\theta}}$ = two discrete points

action of $\tilde{\theta}$

n even



$\pi^{-1}(0)^{\tilde{\theta}}$ = a discrete point

§3.2 multiplicities

$\underset{\text{reduced}}{X^\theta}$ irreducible components $L_1, \cdots, L_m$

$\pi^{-1}(X^\theta)$ irreducible components $\underbrace{\tilde{L}_1, \cdots, \tilde{L}_m}_{|A|}$. $\underbrace{C_1, \cdots, C_n}_{|P|}$

(generically) reduced. could be non-reduced

$$\pi^{-1}(X^\theta) = \sum_{j=1}^m \tilde{L}_j + \sum_{i=1}^n \boxed{a_i} C_i \quad \text{as a divisor.}$$

want to determine a_i

Define $b_i := \#\{\tilde{L}_j \mid \tilde{L}_j \cap C_i \neq \phi\}$ (count how many $\tilde{L}_j$'s intersect C_i)

Prop 4 (multiplicity) If $X^\theta = \mathrm{div}(f) \subset X$ is a principal divisor, then
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \boxed{C^{-1}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

inverse of Cartan matrix

Rmk. $X^\theta \subset X$ is principal modulo two cases