On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

overview + motivation

1) Reminder on Kleinian singularities

2) Anti-Poisson involutions & their fixed loci

3) Preimage of fixed loci

Overview:

$$\begin{array}{ccc} \text{minimal resolution} & & \text{Kleinian singularity.} \\ \tilde{X} & \xrightarrow{\pi} & X \circlearrowleft \theta \quad \text{anti-Poisson involution} \\ \cup & & \cup \\ \pi^{-1}(X^\theta) & & X^\theta \quad \text{Lagrangian subvariety} \end{array}$$

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as subschemes.

Motivation: classification of irreducible $HC(\mathbb{C}\mathfrak{g}, K)$-mod.

- G (simply-connected) simple alg group / $\mathbb{C}$; $\mathfrak{g}$. $\mathcal{N}$ (nilp cone)

  $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ Lie alg involution $\rightsquigarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ $\quad \mathfrak{k} \to K$. connected alg subgrp

  $\theta := -\sigma$ anti-Poisson involution

- $\mathbb{O}$ nilpotent orbit. $\mathbb{O}' \subset \bar{\mathbb{O}}$, $\operatorname{codim}_{\mathbb{C}}(\mathbb{O}', \bar{\mathbb{O}}) = 2$

  $e' \in \mathbb{O}'$, assume e' is normal in $\bar{\mathbb{O}}$

  $\rightsquigarrow$ S' Slodowy slice $\rightsquigarrow$ $S' \cap \bar{\mathbb{O}} \simeq X$. a Kleinian singularity

  $S' = e' + Z_{\mathfrak{g}}(f')$. where $\{e', h', f'\}$ $\mathfrak{sl}_2$-triple.

  $\theta \curvearrowright S' \cap \bar{\mathbb{O}} \rightsquigarrow (S' \cap \bar{\mathbb{O}})^{\theta} \simeq X^{\theta}$. fixed locus

  $(S' \cap \bar{\mathbb{O}})^{\theta} = S' \cap \bar{\mathbb{O}} \cap \mathfrak{p}$ related to $AV(M)$. associated variety of a $HC(\mathbb{C}\mathfrak{g}, K)$-mod M annihilated by the unipotent ideal $J(\mathbb{O})$ s.t. $\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) = 2$

  $\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) \geq 4$. classification done by [Losev & Yu 23]

  $\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) = 2$. classification unknown

Example: $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$

$\sigma : X \longmapsto -X^T$. $\mathfrak{k} = \mathfrak{g}^{\sigma} = \mathfrak{so}(2)$

$\theta = -\sigma : X \longmapsto X^T$, $\mathfrak{p} = \mathfrak{g}^{\theta}$ = symmetric matrices

$\mathbb{O}$ = conj. class of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\bar{\mathbb{O}} = \mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0 \right\} = X$ (Kleinian singularity of type A_1)

$e' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S' = \mathfrak{g}$. $\bar{\mathbb{O}} = S' \cap \bar{\mathbb{O}}$

$X^{\theta} = (S' \cap \bar{\mathbb{O}})^{\theta} = S' \cap \bar{\mathbb{O}} \cap \mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0,\ b = c \right\}$

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- $HC(\mathfrak{g}, K)$-mod : f.g. $\mathcal{U}(\mathfrak{g})$-mod. M. $K \curvearrowright M$ locally finitely & integrates to $K \curvearrowright M$.
  - f.g. $\mathcal{U}(\mathfrak{g})$-mod $\rightsquigarrow$ PBW filtration
  - M $\rightsquigarrow$ good filtration (K-stable)
    - always exists : $X_0 \subset M$ finite set of generators of $\mathcal{U}(\mathfrak{g})$-mod
    - $M_0 :=$ K-rep generated by X_0, M_0 f.dim'l b/c K acts locally finitely
    - $M_n := \mathcal{U}(\mathfrak{g})_n M_0 \rightsquigarrow \dim M_n < \infty$
    - & $M = \bigcup_{n \geq 0} M_n$ as M generated by M_0 as $\mathcal{U}(\mathfrak{g})$-mod

Associated variety $AV(M) := \operatorname{supp}(\operatorname{gr} M) \subset \mathfrak{p}$ (set-theoretic)

($\operatorname{gr} M$: mod over $\operatorname{gr}\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$)

M irreducible $\rightsquigarrow AV(M) \subset \mathcal{N} \cap \mathfrak{p}$

$\mathcal{O}$ nilpotent orbit $\rightsquigarrow J(\mathcal{O})$ unipotent ideal associated with $\mathcal{O}$

$J(\mathcal{O}) = \ker[\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{A}_0]$ ($\mathcal{A}_0$: canonical quantization of $\mathbb{C}[\mathcal{O}]$)

consider irreducible M annihilated by $J(\mathcal{O}) \rightsquigarrow AV(M) \subset \bar{\mathcal{O}} \cap \mathfrak{p}$

- $\operatorname{codim}(\partial\bar{\mathcal{O}}, \bar{\mathcal{O}}) \geq 4$ ($\bar{\mathcal{O}}$ terminal) $\rightsquigarrow$ [Vogan 91] $AV(M) = \overline{\mathcal{O}_K}$ is irreducible

  $\overline{\mathcal{O}_K}$ = closure of single K-orbit in $\mathcal{O} \cap \mathfrak{p}$

  [Losev & Yu 23], classified irreducible $HC(\mathfrak{g}, K)$-mod annihilated by $J(\mathcal{O})$ s.t. $AV(M) = \overline{\mathcal{O}_K}$.

- $\operatorname{codim}(\partial\bar{\mathcal{O}}, \bar{\mathcal{O}}) = 2$. classification unknown,

  $AV(M)$. may not be irreducible

  $AV(M) \subset \bar{\mathcal{O}} \cap \mathfrak{p}$

  S', slice to $e' \in \mathcal{O}'$, $\operatorname{codim}_{\bar{\mathcal{O}}} \mathcal{O}' = 2$

  $S' \cap \bar{\mathcal{O}} \cap \mathfrak{p} = (S' \cap \bar{\mathcal{O}})^{\theta} = X^{\theta}$

§1. Reminder on Kleinian singularities

$\Gamma \subset SL_2(\mathbb{C})$. finite subgroup

<u>Kleinian singularity</u>: $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u,v]^{\Gamma}$

<u>Example</u>: $\Gamma = \{\pm I_2\}$

$$\mathbb{C}[u,v]^{\Gamma} = \mathbb{C}[x = u^2,\ y = v^2,\ z = uv] = \mathbb{C}[x,y,z]/(xy - z^2)$$

<u>Fact (Klein)</u>: $X = \mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (single relation) with an isolated singularity at 0.

<u>minimal resolution</u>: $\pi : \tilde{X} \longrightarrow X$, projective & birational. ($\tilde{X}$ smooth)

exceptional locus: $\pi^{-1}(0)_{red} = C_1 \cup \cdots \cup C_n$, $C_i \simeq \mathbb{P}^1$ (irreducible component)

Dual graph of $\pi^{-1}(0)_{red}$: $C_i \rightsquigarrow \bullet\ i$

$C_i \cap C_j \neq \phi \rightsquigarrow$ i •——• j

<u>Fact (Du Val)</u>: Dual graphs of $\pi^{-1}(0)_{red}$ $\xleftrightarrow{\text{1-to-1}}$ ADE Dynkin diagrams

<u>Rem</u>: $\pi^{-1}(0)$ is not reduced in general:

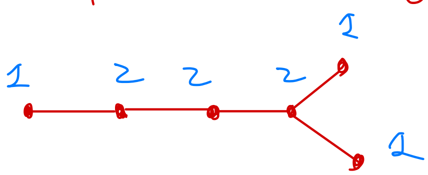
$\mathfrak{g}$ simple Lie alg of types ADE, simple root system $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$

δ - unique maximal root $\delta = \sum_{i=1}^{n} m_i \alpha_i$

Then $\pi^{-1}(0) = \sum_{i=1}^{n} m_i C_i$ as a divisor [Artin]

Type A_n. $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\delta = \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_n \rightsquigarrow \pi^{-1}(0)$ reduced

Type D_n $(\delta_1 \cdots \delta_n) = (1, 2 \cdots 2, 1, 1) \rightsquigarrow \pi^{-1}(0)$ not reduced



Examples:

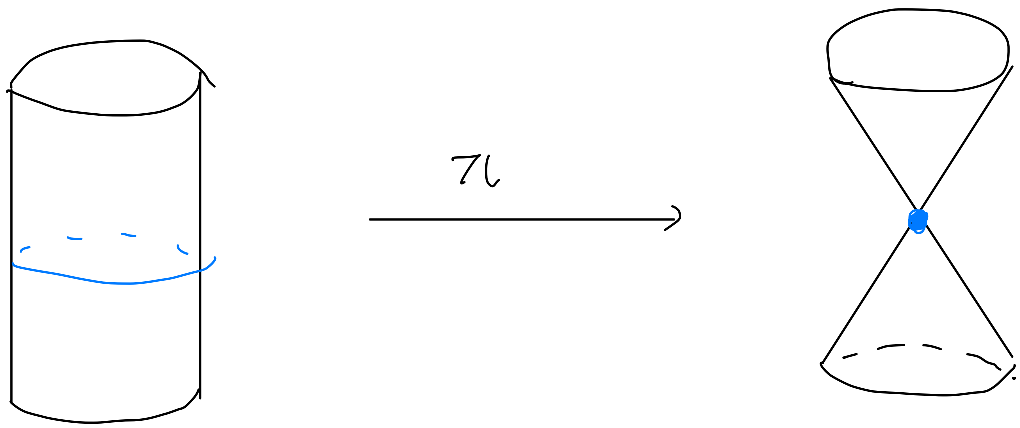
1) A_1

$\pi^{-1}(0) = \mathbb{P}^1$

$\downarrow$ dual graph

$\bullet$

A_1



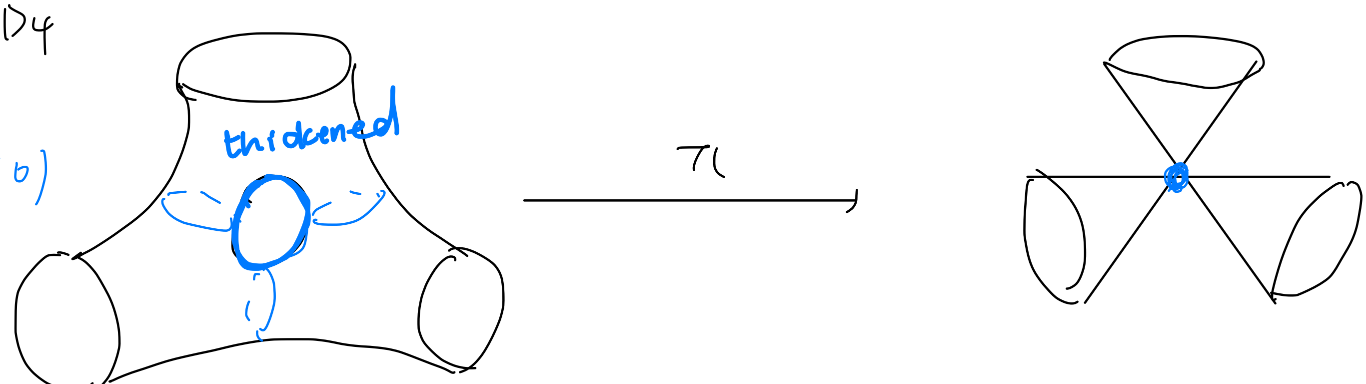
$T^*\mathbb{P}^1$

$X = (xy - z^2 = 0)$

Similarly: Type A_n $\pi^{-1}(0)$:

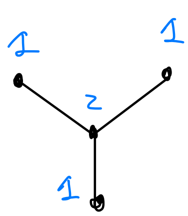
2) D_4

$\pi^{-1}(0)$



$X = (x^3 + xy^2 + z^2 = 0)$

dual graph:



D_4 Dynkin diagram

§2. Anti-Poisson involutions

$X = \mathbb{C}^2/\Gamma$, Kleinian singularity

$\mathbb{C}[u, v]$ graded Poisson alg

$\cup$

$\mathbb{C}[X] = \mathbb{C}[u, v]^{\Gamma}$ graded Poisson subalg.

<u>Example</u>: Type A_n: $\mathbb{C}[X] = \mathbb{C}[x, y, z] / (xy - z^{n+1})$

$$\{x, y\} = (n+1)^2 z^n$$
$$\{x, z\} = (n+1) x$$
$$\{y, z\} = -(n+1) y$$

<u>Def</u>: An anti-Poisson involution of $X \cong \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta: \mathbb{C}[X] \to \mathbb{C}[X]$ s.t.

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\}, \quad \forall f_1, f_2 \in \mathbb{C}[X]$$

<u>Example</u>: Type A_n: $\theta: \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

$$x \longmapsto y$$
$$y \longmapsto x$$
$$z \longmapsto z$$

<u>Def (scheme-theoretic fixed locus)</u>

$$X^{\theta} := \operatorname{Spec} \mathbb{C}[X]/I \quad \text{where} \quad I = (\theta(f) - f \mid f \in \mathbb{C}[X])$$

Example: (continued)

$$X^{\theta} = \mathrm{Spec}\, \mathbb{C}[x,y,z]/(xy - z^{n+1}, x-y) \simeq \mathrm{Spec}\, \mathbb{C}[x,z]/(x^2 - z^{n+1})$$

$\downarrow$ reduced

$$= \begin{cases} \text{union of two } \mathbb{A}^1, & n \text{ odd} \\ \text{cusp}, & n \text{ even} \end{cases}$$

Prop 2 (H.) Classification of θ

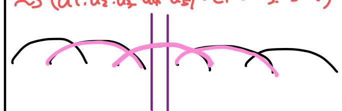
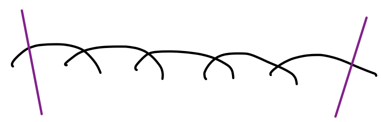
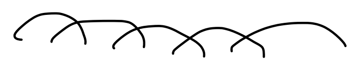
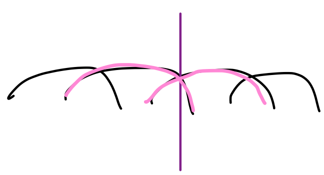
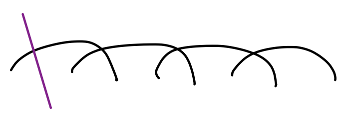
There are finitely many anti-Poisson involution on $X = \mathbb{C}^2/\Gamma$ up to conjugation by Poisson automorphism. $\longleftrightarrow$ $N_{GL_2(\mathbb{C})}(\Gamma)^- / N_{SL_2(\mathbb{C})}(\Gamma)$ up to conj.

Prop 2 (H.) Description of X^{θ}

The scheme-theoretic fixed locus X^{θ} is reduced.

If $X^{\theta} \neq \{0\}$, each irreducible component of X^{θ} is either an $\mathbb{A}^1$ or a cusp.

Example: Anti-Poisson involution for type A_n Kleinian singularities

Lie alg involution	Type I $\sigma: X \longmapsto -X^t$	Type II $\sigma = Ad(I_{k,l})$	Type III Not seen unless in $sl_2(\mathbb{C})$, where $\sigma = id$.
A_n, n odd $xy - z^{n+1} = 0$ X^θ	$x \longleftrightarrow y$ 	$z \longmapsto -z$ 	$x \longmapsto -x$ $y \longmapsto -y$ $z \longmapsto -z$
$\pi^{-1}(X^\theta)$	$(b_1\ b_2\ b_3\ b_4\ b_5) = (0, 0, 2, 0, 0)$ $\rightsquigarrow (a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 2, 1)$  non-reduced components		
A_n, n even $xy - z^{n+1} = 0$ X^θ	$x \longleftrightarrow y$ 	$x \longmapsto -x$ $z \longmapsto -z$ 	N/A.
$\pi^{-1}(X^\theta)$			

§3. Preimages

X Kleinian singularity, θ: anti-Poisson involution

$\pi : \tilde{X} \longrightarrow X \simeq \mathbb{C}^2/\Gamma$ minimal resolution

$0 \in X^\theta \rightsquigarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$. but there are more

§3.1 lift

Main Thm (H.) There exists a unique anti-symplectic involution $\tilde{\theta} : \tilde{X} \to \tilde{X}$ s.t. $\pi \circ \tilde{\theta} = \theta \circ \pi$. We call $\tilde{\theta}$ a lift of θ

Proof of Main Thm is through the realization of Kleinian singularities as Nakajima quiver varieties

Why the lift helps?

$\tilde{X}$ smooth $\rightsquigarrow$ $\tilde{X}^{\tilde{\theta}}$ smooth Lagrangian

Claim: $\pi^{-1}(X^\theta)_{red} = \pi^{-1}(0)_{red} \cup \boxed{\tilde{X}^{\tilde{\theta}}}$

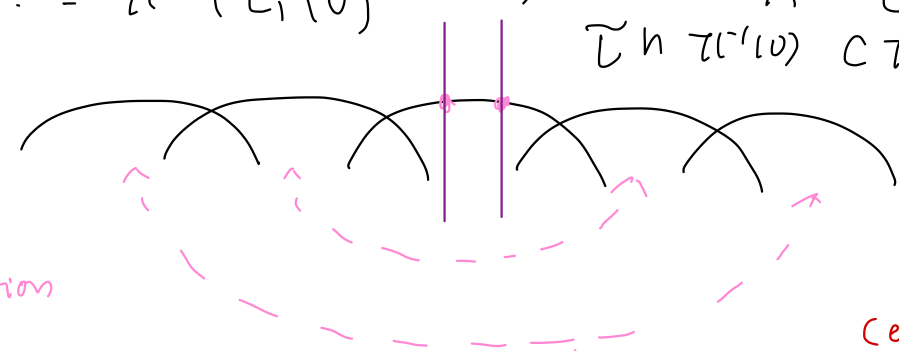
no intersection, no cusp

$L \subset X^\theta$, irreducible component

Define $\tilde{L}_i := \overline{\pi^{-1}(L_i \setminus 0)} \rightsquigarrow \tilde{L} \simeq \mathbb{A}^1 \subset$ Claim)

$\tilde{L} \cap \pi^{-1}(0) \subset \pi^{-1}(0)^{\tilde{\theta}}$

Example:



$\tilde{\theta}$ action

Type I.

in Type A_n, n odd

(even case skipped)

§3-2 multiplicities

X^0 . irreducible components $L_1, \cdots L_m$ ('A' or cusp)

$\pi^{-1}(X^0)$ irreducible components $\tilde{L}_1 \cdots \tilde{L}_m$, $C_1 \cdots C_n$

$\tilde{L}_1 \cdots \tilde{L}_m$: 'A' reduced

$C_1 \cdots C_n$: 'P' could be non-reduced.

$$\pi^{-1}(X^0) = \sum_{j=1}^{m} \tilde{L}_j + \sum_{i=1}^{n} a_i C_i \quad \text{as a divisor}$$

a_i → determ a_i

Define $b_i := \# \{ \tilde{L}_j \mid \tilde{L}_j \cap C_i \neq \phi \}$.

Prop 3 (H.) multiplicities If $X^0 \subset X$ is a principal divisor.

→ almost always true (modulo A_n n odd type III, n even type II)

then

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = C^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

C^{-1}: inverse of Cartan matrix