On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

Mengwei Hu

Yale University

AWM Research Symposium

Geometric and topological aspects of mathematical physics and representation theory

May 16-18, 2025

arXiv: 2504.08717

Overview:

minimal resolution		Kleinian singularity	
$\widetilde{X}$	$\xrightarrow{\pi}$	$X \circlearrowleft \theta$	anti-Poisson involution
$\cup$		$\cup$	
$\pi^{-1}(X^\theta)$		X^θ	Lagrangian subvariety

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as schemes.

Scenario: The singularity of a subregular nilpotent element in the nilpotent cone of a simple Lie algebra is a Kleinian singularity. Interesting θ comes from Lie algebra involutions.

Motivation: Classify (certain) irreducible Harish-Chandra $(\mathfrak{g}, K)$-modules in geometric terms. Roughly speaking, Supp(HC modules) $\approx X^\theta$.

Example

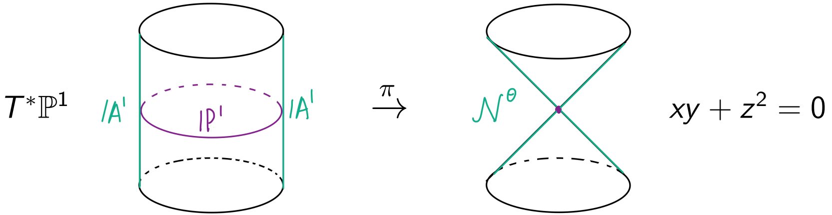
Nilpotent cone of $\mathfrak{sl}_2(\mathbb{C})$: $\mathcal{N} = \left\{ \begin{pmatrix} z & x \\ y & -z \end{pmatrix} \middle| \; xy + z^2 = 0 \right\}$

Lie algebra (anti-)involution $\theta\colon M \mapsto M^t$ anti-Poisson involution

The fixed point locus $\mathcal{N}^\theta = \left\{ \begin{pmatrix} z & x \\ y & -z \end{pmatrix} \middle| \; xy + z^2 = 0, \; x = y \right\}$

(symmetric matrices in $\mathcal{N}$)

Springer resolution $\pi\colon \widetilde{\mathcal{N}} = T^*\mathbb{P}^1 \to \mathcal{N}$. Study preimage $\pi^{-1}(\mathcal{N}^\theta)$.



Kleinian singularity of type A_1

Kleinian singularities

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ finite subgroup. There are five conjugacy classes.

- **(A_n)** cyclic group of order $n+1$. $xy - z^{n+1} = 0$
- **(D_n)** binary dihedral group of order $4(n-2)$. $x^{n-1} + xy^2 + z^2 = 0$
- **(E_6)** binary tetrahedral group of order 24. $x^4 + y^3 + z^2 = 0$
- **(E_7)** binary octahedral group of order 48. $x^3y + y^3 + z^2 = 0$
- **(E_8)** binary icosahedral group of order 120. $x^5 + y^3 + z^2 = 0$

Definition

The Kleinian singularity attached to Γ is $X := \mathbb{C}^2/\Gamma = \mathrm{Spec}\,\mathbb{C}[u,v]^\Gamma$.

Example: $\Gamma = \{\pm I_2\}$, we have $\mathbb{C}[u,v]^\Gamma =$ even degree polynomials $= \mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x,y,z]/(xy - z^2)$.

Fact (Klein): $\mathbb{C}^2/\Gamma$ can be viewed as a hypersurface in $\mathbb{C}^3$ with an isolated singularity at 0.

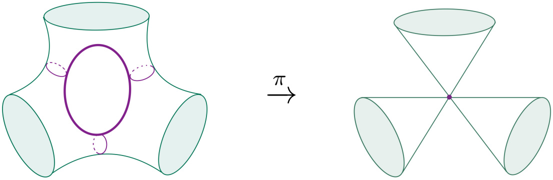
McKay correspondence: $\{\Gamma\} \xleftrightarrow{1-1} \{\text{ADE Dynkin diagrams}\}$

Minimal resolutions

McKay correspondence: Kleinian singularities are in bijection with ADE Dynkin diagrams.

Minimal resolution $\pi : \widetilde{X} \to X = \mathbb{C}^2/\Gamma$, projective, birational, "minimal"
Exceptional fiber $\pi^{-1}(0) = C_1 \cup \cdots \cup C_n, \; C_i \simeq \mathbb{P}^1$, with pairwise transversal intersection according dually to a Dynkin diagram.
More precisely, replace each C_i by a vertex i and draw an edge between vertices i, j if C_i intersects with $C_j \rightsquigarrow$ a Dynkin diagram of types ADE.

Example: Kleinain singularity of type D_4 : $x^3 + xy^2 + z^2 = 0$



Connection to Lie theory

Singularity of subregular nilpotent element

Take $\mathfrak{g}$ a simple Lie algebra of types ADE, $\mathcal{N}$ its nilpotent cone. Take e a subregular nilpotent element and pick an $\mathfrak{sl}_2$-triple $\{e, h, f\}$. Consider the Slodowy slice $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$, then $S \cap \mathcal{N} \simeq \mathbb{C}^2/\Gamma$ is a Kleinian singularity of the corresponding type.

Ways to construct the minimal resolution:

- iterated blow-ups
- quiver varieties and GIT quotients
- base change under Springer resolution $\widetilde{\mathcal{N}} \times_{\mathcal{N}} (S \cap \mathcal{N}) \to S \cap \mathcal{N}$

(exceptional fiber = Springer fiber of the subregular nilpotent element)

Example: $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, Lie algebra of type A_n, Springer fiber of the subregular nilpotent element $(n, 1)$ consists of n-$\mathbb{P}^1$'s.

Anti-Poisson involutions

Set $X := \mathbb{C}^2/\Gamma$. The algebra of functions $\mathbb{C}[X] = \mathbb{C}[u,v]^\Gamma$ is a graded (*by degree of polynomials in* u, v) Poisson algebra with Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

Definition

An **anti-Poisson involution** of a Kleinian singularity $X = \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\}, \ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

Example: Type A_n Kleinian singularity $\mathbb{C}[x,y,z]/(xy - z^{n+1})$. The Poisson brackets are given by

$$\{x,y\} = (n+1)^2 z^n,\ \{x,z\} = (n+1)x,\ \{y,z\} = -(n+1)y.$$

One verifies $\theta\colon x \mapsto y,\ y \mapsto x,\ z \mapsto z$ is an anti-Poisson involution.

The fixed point loci

Definition

$X = \mathbb{C}^2/\Gamma$ with anti-Poisson involution θ. The **fixed point locus** is $X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$.

Example: Type A_n singularity $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1})$ with θ swapping $x \leftrightarrow y$. We have

$X^\theta = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1}, x - y) \simeq \operatorname{Spec} \mathbb{C}[x, z]/(x^2 - z^{n+1})$,

which is a union of two $\mathbb{A}^1$'s when n is odd, a cusp when n is even.

Proposition 1 (H.)

- There are finitely many anti-Poisson involutions on $X = \mathbb{C}^2/\Gamma$ up to conjugation by graded Poisson automorphisms.
- The fixed point locus X^θ is reduced.
- Each irreducible component of X^θ is either $\mathbb{A}^1$ or a cusp.

Examples and connections

Example: Anti-Poisson involutions for type A_n Kleinian singularities.

	case I	case II	case III
A_n, n odd $xy - z^{n+1} = 0$	$x \mapsto y$ $y \mapsto x$ $z \mapsto z$	$x \mapsto x$ $y \mapsto y$ $z \mapsto -z$	$x \mapsto -x$ $y \mapsto -y$ $z \mapsto -z$
A_n, n even $xy - z^{n+1} = 0$	$x \mapsto y$ $y \mapsto x$ $z \mapsto z$	$x \mapsto -x$ $y \mapsto y$ $z \mapsto -z$	
involution of $\mathfrak{sl}_{n+1}$	$M \mapsto -M^t$	$\mathrm{Ad}(I_{p,q})$	

• Recall $S \cap \mathcal{N}$ is Kleinian singularity. Involution of $\mathfrak{g}$ restricts to $S \cap \mathcal{N}$.
Interesting anti-Poisson involutions come from Lie algebra involutions.

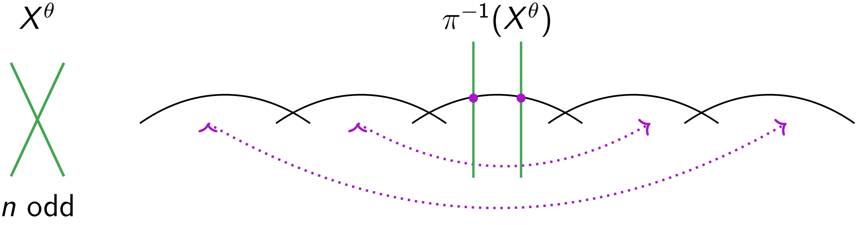
• irreducible components of X^θ are K-orbits in the symmetric space.
E.g. $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $K = \mathrm{SO}(n+1) \curvearrowright \{\text{symmetric matrices in } \mathcal{N}\}$.
Regular nilpotent elements break into two orbits when n is odd, and form a single orbit when n is even.

Remark: D_n, E_6, two cases; E_7, E_8, one case.

Preimage of fixed point loci under minimal resolutions

$\pi\colon \widetilde{X} \to X = \mathbb{C}^2/\Gamma$ denotes the minimal resolution. We would like to describe the preimage $\pi^{-1}(X^\theta)$. $0 \in X^\theta \Rightarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$.

Example: Type A_n singularity with θ swapping $x \leftrightarrow y$.



Straight green line is $\mathbb{A}^1$, curly black line is $\mathbb{P}^1$.

Idea: Construct a lift of θ to $\widetilde{X}$, study how $\tilde{\theta}$ acts on $\pi^{-1}(0)$.

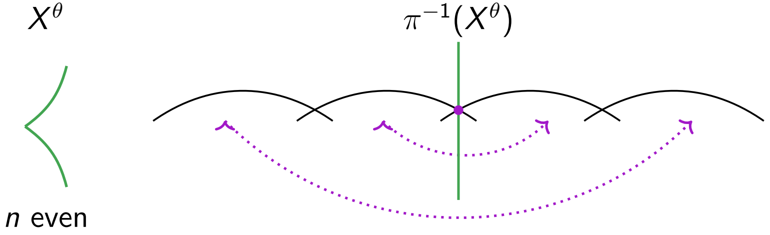
Takeaways: • Each irreducible component of $\pi^{-1}(X^\theta)$ is $\mathbb{A}^1$ or $\mathbb{P}^1$.

• $\mathbb{A}^1$ intersects $\mathbb{P}^1$ at the isolated $\tilde{\theta}$-fixed points in $\pi^{-1}(0)$.

Preimage of fixed point loci under minimal resolutions

$\pi\colon \widetilde{X} \to X = \mathbb{C}^2/\Gamma$ denotes the minimal resolution. We would like to describe the preimage $\pi^{-1}(X^\theta)$. $0 \in X^\theta \Rightarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$.

Example: Type A_n singularity with θ swapping $x \leftrightarrow y$.



Straight green line is $\mathbb{A}^1$, curly black line is $\mathbb{P}^1$.

Idea: Construct a lift of θ to $\widetilde{X}$, study how $\tilde{\theta}$ acts on $\pi^{-1}(0)$.

Takeaways: • Each irreducible component of $\pi^{-1}(X^\theta)$ is $\mathbb{A}^1$ or $\mathbb{P}^1$.

• $\mathbb{A}^1$ intersects $\mathbb{P}^1$ at the isolated $\tilde{\theta}$-fixed points in $\pi^{-1}(0)$.

Lift anti-Poisson involutions

$\pi : \widetilde{X} \to X$, minimal resolution, and θ an anti-Poisson involution of X.

Theorem (H.)

There exists a unique anti-symplectic involution $\tilde{\theta} : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \tilde{\theta} = \theta \circ \pi$. It can be constructed explicitly via quiver varieties.

Fact: $\widetilde{X}$ smooth $\Rightarrow \widetilde{X}^{\tilde{\theta}}$ is smooth Lagrangian (no intersection, no cusp).
With the lift $\tilde{\theta}$, one can describe $\pi^{-1}(X^{\theta})_{\text{red}} = \pi^{-1}(0)_{\text{red}} \cup \widetilde{X}^{\tilde{\theta}}$.

To determine $\pi^{-1}(X^{\theta})$, need further analysis.
Set $m := \#$ of irreducible components of X^{θ}.

Method: Write $\pi^{-1}(X^{\theta}) = \sum_{j=1}^{m} 1 \cdot L_j + \sum_{i=1}^{n} a_i C_i$ as a divisor, with $L_j \simeq \mathbb{A}^1,\ C_i \simeq \mathbb{P}^1$. **Fact:** $\pi^{-1}(X^{\theta})$ is reduced $\Leftrightarrow a_i = 1$.

Define $b_i := \#$ of $\mathbb{A}^1$'s that a $\mathbb{P}^1$ intersects with.

Multiplicities

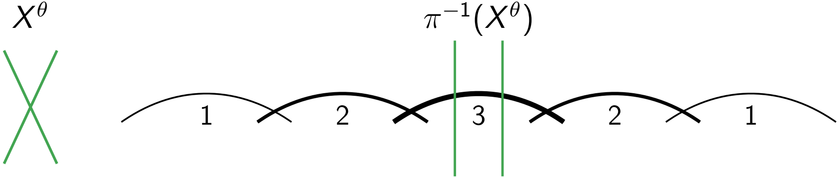
Proposition 2 (H.)

If $X^\theta \subset X$ is a principal divisor, then

$$(a_1, \cdots, a_n)^t = \mathcal{C}^{-1}(b_1, \cdots, b_n)^t,$$

where $\mathcal{C}$ is the corresponding Cartan matrix of types ADE.

Example: Type A_5 singularity with θ swapping $x \leftrightarrow y$.

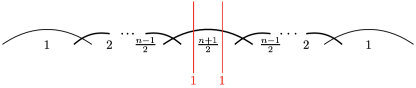


$(b_1, b_2, b_3, b_4, b_5) = (0, 0, 2, 0, 0)$

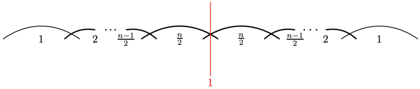
$(a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 2, 1)$ not reduced

Preimages in type A

Straight line is $\mathbb{A}^1$. Curly line is $\mathbb{P}^1$. The number next to each component is its multiplicity. The $\tilde{\theta}$-fixed components are colored in red.

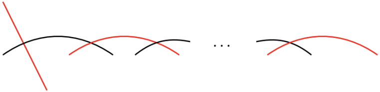


A_n, n odd, case I



A_n, n even, case I

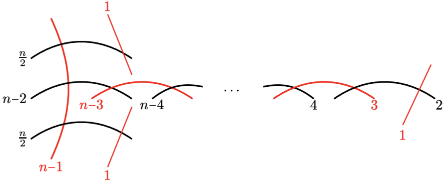
A_n, n odd, case II
(reduced)



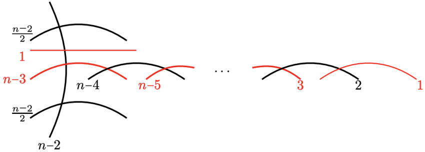
A_n, n even, case II
(reduced)

Preimages in type D

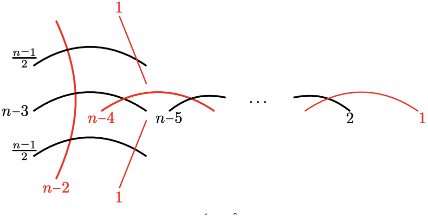
Straight line is $\mathbb{A}^1$. Curly line is $\mathbb{P}^1$. The number next to each component is its multiplicity. The $\tilde{\theta}$-fixed components are colored in red.



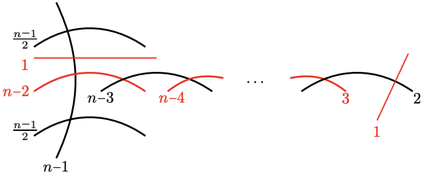
D_n, n even, case I



D_n, n even, case II



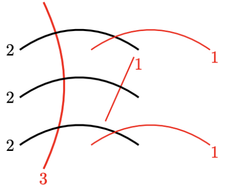
D_n, n odd, case I



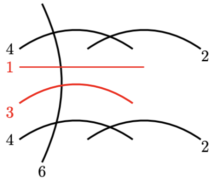
D_n, n odd, case II

Preimages in type E

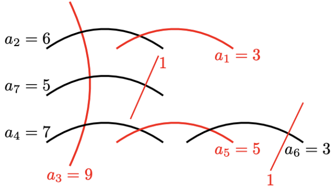
Straight line is $\mathbb{A}^1$. Curly line is $\mathbb{P}^1$. The number next to each component is its multiplicity. The $\tilde{\theta}$-fixed components are colored in red.



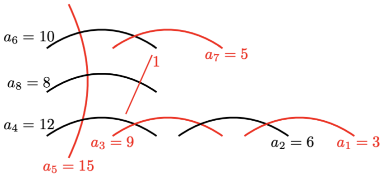
E_6, case I



E_6, case II



E_7



E_8

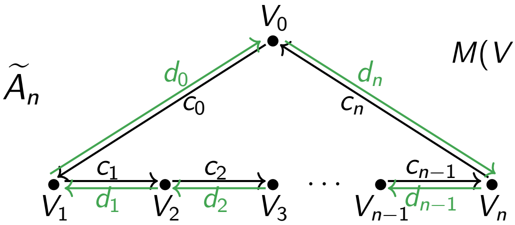
Proof of Main Theorem

Theorem (H.)

There exists a unique anti-symplectic involution $\tilde{\theta}$ of $\widetilde{X}$ that lifts θ.

Idea of proof: realize Kleinian singularity as Nakajima quiver variety.

Example: Kleinian singularity of type A_n: $xy - z^{n+1}$ as a quiver variety.



$$\dim V_i = 1$$

$$M(V) := T^*(\bigoplus_{i=0}^n \mathrm{Hom}(V_i, V_{i+1}))$$

$$G := \prod_{i=0}^n \mathrm{GL}(V_i) \curvearrowright M(V)$$

$$\mu : M(V) \to \oplus_{i=0}^n \mathfrak{gl}(V_i)$$

$$\mu_i = c_{i-1}d_{i-1} - d_i c_i$$

We have $\mu^{-1}(0)//G \simeq \mathrm{Spec}\,\mathbb{C}[x, y, z]/(xy - z^{n+1})$ with

$$x := c_n c_{n-1} \cdots c_1 c_0,\ \ y := d_0 d_1 \cdots d_{n-1} d_n,\ \ z := c_0 d_0.$$

The lift of θ: $x \leftrightarrow y$ is $\tilde{\theta}$: $c_i \leftrightarrow d_{n-i}$.

Thank you!