# On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

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## Overview

| minimal resolution | | Kleinian singularity | |
|---|---|---|---|
| $\tilde{X}$ | $\xrightarrow{\pi}$ | $X = \mathbb{C}^2/\Gamma \circlearrowleft \theta$ | anti-Poisson involution |
| $\cup$ | | $\cup$ | |
| $\pi^{-1}(X^\theta)$ | | $X^\theta$ | Lagrangian subvariety |

**Goal: Describe $X^\theta$ and $\pi^{-1}(X^\theta)$ as schemes.**

**Components? Intersection Pattern? Reduced?**

## Kleinian singularities

Kleinian singularities are the only symplectic singularities in dim 2.

- $X := \mathbb{C}^2/\Gamma = \operatorname{Spec}\mathbb{C}[u,v]^\Gamma$, $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ finite subgroup.
- $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation) with the only singularity at 0.

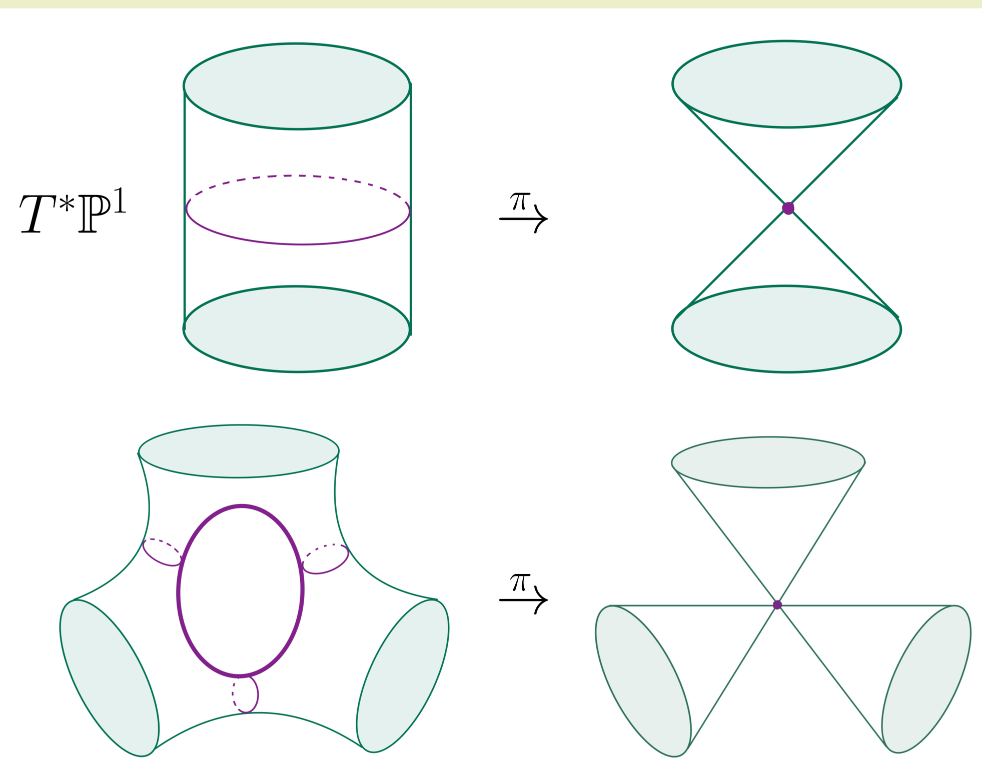
$(A_n)$ $\mathbb{Z}/(n+1)\mathbb{Z} \rightsquigarrow x = u^{n+1},\ y = v^{n+1},\ z = uv \rightsquigarrow xy - z^{n+1} = 0.$

$(D_n)$ $x^{n-1} + xy^2 + z^2 = 0,\ n \geqslant 4.$

$(E_6)\dots(E_7)\dots(E_8)\dots$

- **McKay correspondence:** $\{\Gamma\} \xleftrightarrow{1-1} \{\text{ADE Dynkin diagrams}\}$.
- Minimal resolution $\pi\colon \tilde{X} \to X = \mathbb{C}^2/\Gamma$.
- Exceptional fiber $\pi^{-1}(0) = C_1 \cup \cdots \cup C_n$, $C_i \simeq \mathbb{P}^1$, pairwise transversal intersection according to a Dynkin diagram of types ADE.

## Example




$A_1$ singularity $xy - z^2 = 0$

$D_4$ singularity $x^3 + xy^2 + z^2 = 0$

## Anti-Poisson involutions & fixed point loci

Set $X = \mathbb{C}^2/\Gamma$. The algebra of functions $\mathbb{C}[X] = \mathbb{C}[u,v]^\Gamma$ is a graded (by degree of polynomials in $u, v$) Poisson algebra with Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

**Definition:** An **anti-Poisson involution** of $X = \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\},\ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

**Definition:** The **fixed point locus** is $X^\theta := \operatorname{Spec}\mathbb{C}[X]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$.

### Proposition 1 (H., 2025)

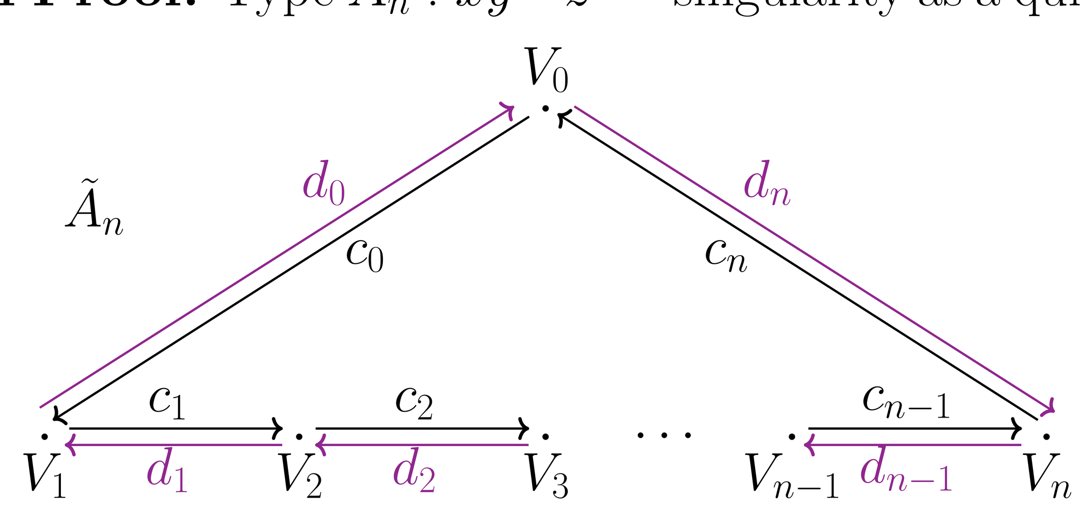
- There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by graded Poisson automorphisms.
- The fixed point locus $X^\theta$ is reduced.
- Each irreducible component of $X^\theta$ is either $\mathbb{A}^1$ or a cusp.

## Main Theorem

### Theorem (H., 2025)

There exists a unique anti-symplectic involution $\tilde{\theta}: \tilde{X} \to \tilde{X}$ such that $\pi \circ \tilde{\theta} = \theta \circ \pi$.

**Sketch of Proof:** Type $A_n : xy - z^{n+1}$ singularity as a quiver variety.




$\dim V_i = 1$. Moment map $\mu_i = c_{i-1}d_{i-1} - d_i c_i = 0$.

$$x = c_n c_{n-1} \cdots c_1 c_0,\ y = d_0 d_1 \cdots d_{n-1} d_n,\ z = c_0 d_0.$$

The lift of $\theta\colon x \leftrightarrow y$ is $\tilde{\theta}\colon c_i \leftrightarrow d_{n-i}$.

**Remark:** Types DE similar, but harder to pick out the generators $x, y, z$ as trace functions because $\dim V_i \neq 1$.

## Preimages

**Takeaways:**

- $\tilde{X}$ smooth $\Rightarrow \tilde{X}^{\tilde{\theta}}$ is smooth Lagrangian. $\pi^{-1}(X^\theta) = \tilde{X}^{\tilde{\theta}} \cup \pi^{-1}(0)$.
- Each irreducible component of $\pi^{-1}(X^\theta)$ is $\mathbb{A}^1$ or $\mathbb{P}^1$.
- $\mathbb{A}^1$ intersects a unique $\mathbb{P}^1$ at an isolated $\tilde{\theta}$-fixed point of $\pi^{-1}(0)$.

**Scheme structure:** Write $\pi^{-1}(X^\theta) = \Sigma_{j=1}^m L_j + \Sigma_{i=1}^n a_i C_i$ as a divisor, where $L_j \simeq \mathbb{A}^1$, $C_i \simeq \mathbb{P}^1$. Set $b_i :=$ # of $\mathbb{A}^1$'s that a $\mathbb{P}^1$ intersects with.

### Proposition 2 (H., 2025)

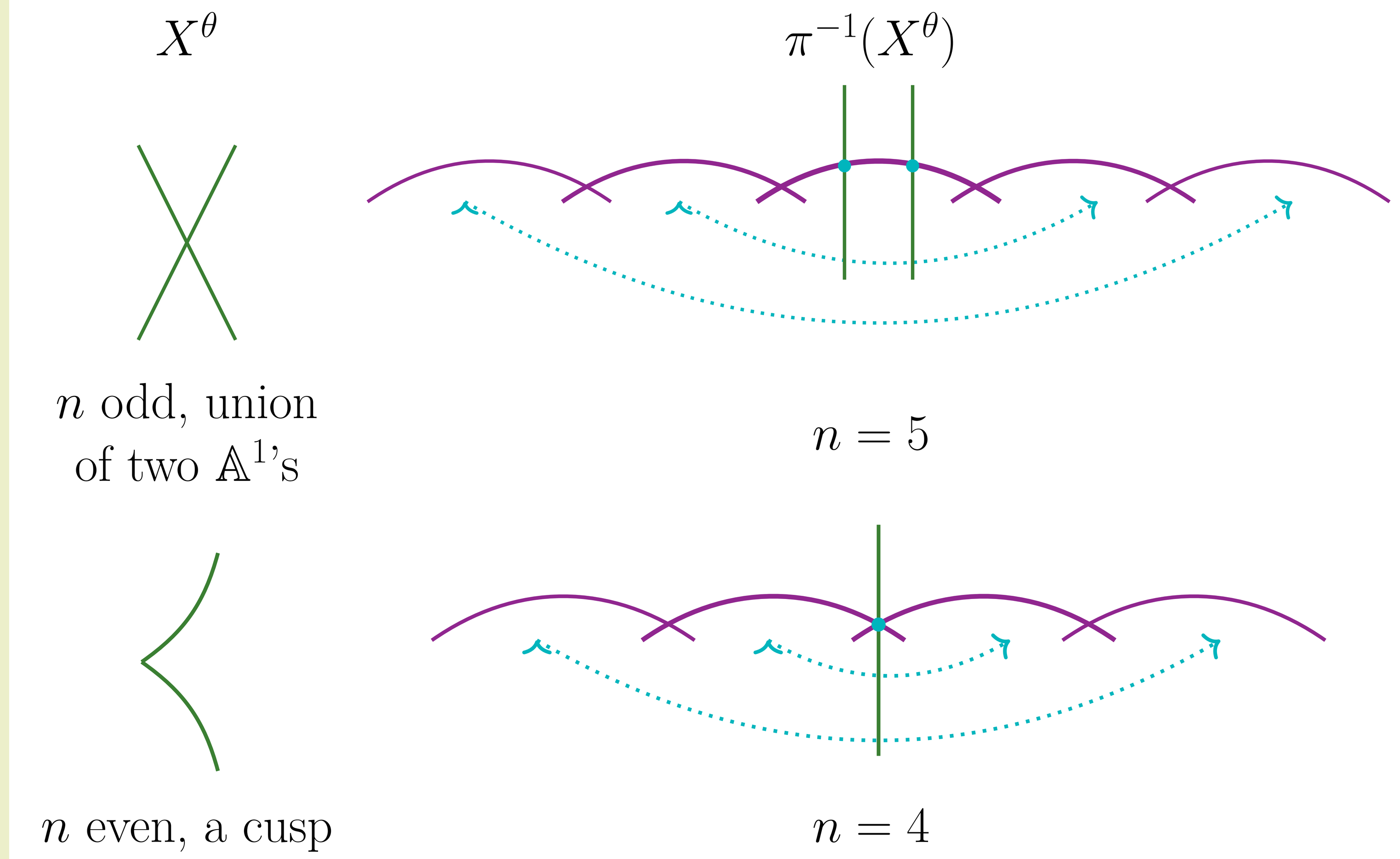
If $X^\theta \subset X$ is a principal divisor, then

$$(a_1, \cdots, a_n)^t = \mathcal{C}^{-1}(b_1, \cdots, b_n)^t,$$

where $\mathcal{C}$ is the corresponding Cartan matrix of types ADE.

## Example

Type $A_n$: $xy - z^{n+1} = 0$ singularity with $\theta$ swapping $x \leftrightarrow y$. Fixed point locus $X^\theta = \operatorname{Spec}\mathbb{C}[x,z]/(x^2 - z^{n+1})$.




**Straight green line is $\mathbb{A}^1$, curly purple line is $\mathbb{P}^1$. The action of $\tilde{\theta}$ on $\pi^{-1}(0)$ is given by the dotted blue arrows.**

**Multiplicities:** $n = 5$, $(b_1, b_2, b_3, b_4, b_5) = (0, 0, 2, 0, 0)$.
Prop 2 $\Rightarrow (a_1, a_2, a_3, a_4, a_5) = (1, 2, 3, 2, 1)$.
$\pi^{-1}(X^\theta)$ is not reduced. Non-reduced components in thicker lines.