Geometry of anti-Poisson involution in the deformations and resolutions of Kleinian singularities

Mengwei Hu

Yale University

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Overview: Kleinian singularity $X \circlearrowleft \theta$, anti-Poisson involution

$$
\begin{array}{ccccccccc}
 & & & \text{universal resolution} & & \text{universal deformation} & & & \\
\pi^{-1}(\mathcal{X}_\lambda^\theta) & \subset & \mathcal{Y}_{\tilde\lambda} & \subset & \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} & \supset & \mathcal{X}_\lambda & \supset & \mathcal{X}_\lambda^\theta \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & \tilde\lambda & \in & \mathfrak{h} & \longrightarrow & \mathfrak{h}/W & \ni & \lambda & &
\end{array}
$$

$\mathcal{X}$ satisfies $\mathcal{X}_0 = X$, $\operatorname{gr}\mathbb{C}[\mathcal{X}_\lambda] = \mathbb{C}[X]$, and universal property.
The restriction $\pi|_{\mathcal{Y}_{\tilde\lambda}} : \mathcal{Y}_{\tilde\lambda} \to \mathcal{X}_\lambda$ is a resolution of singularity.

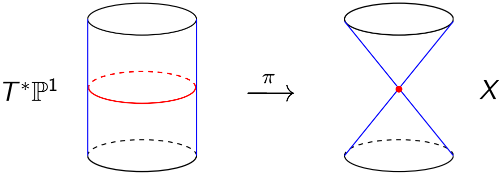
Assume compatibility of λ and θ.

Goal: Describe $\mathcal{X}_\lambda^\theta$ and $\pi^{-1}(\mathcal{X}_\lambda^\theta)$ as schemes.

Motivation: Classify Harish-Chandra modules over quantizations of nilpotent orbits. Supp(HC modules) $\approx X^\theta$. Filtered quantizations and Poisson deformations are both parameterized by $\mathfrak{h}/W$. Use geometric info from Poisson deformation to study filtered quantizations.

An Example

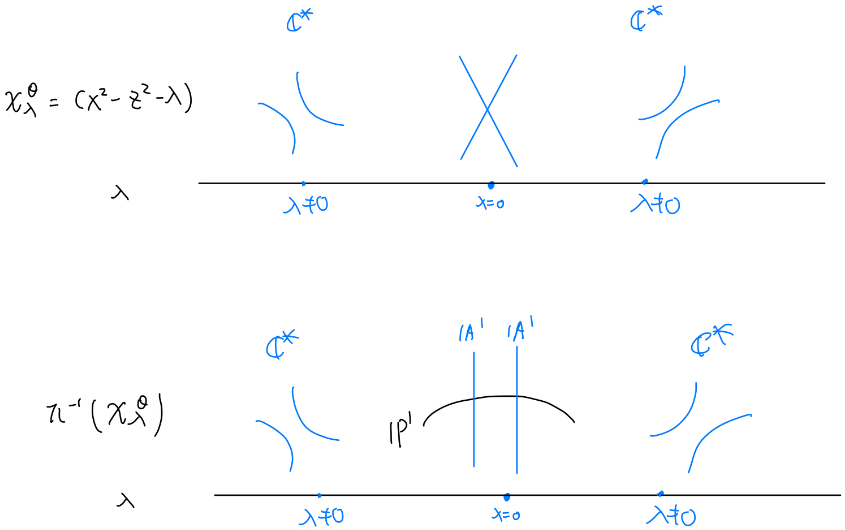
Type A_1 Kleinian singularity: $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$
Anti-Poisson involution $\theta : x \leftrightarrow y$



X^θ is union of two $\mathbb{A}^1$'s; $\pi^{-1}(X^\theta)$ is the union of two $\mathbb{A}^1$ and a $\mathbb{P}^1$.

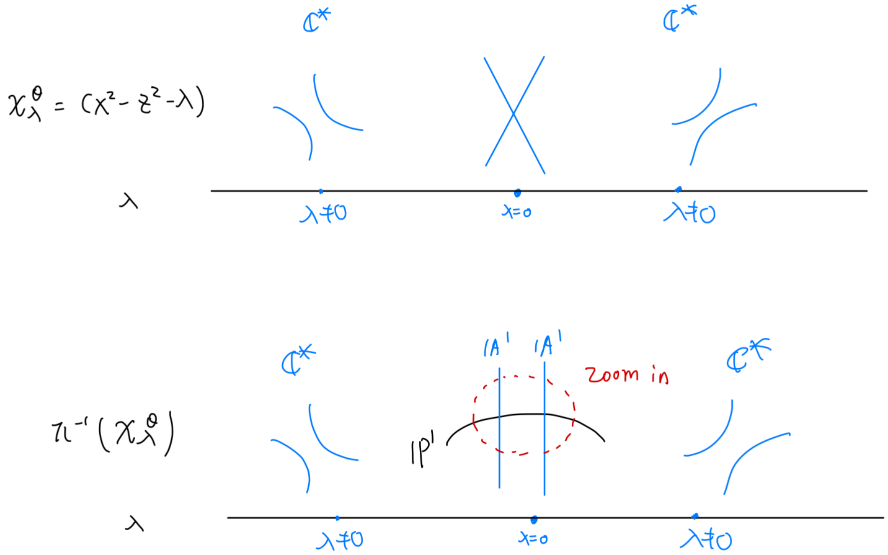
Poisson deformation, $\mathcal{X}_\lambda = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2 - \lambda) \curvearrowleft \theta$
$\mathcal{X}_\lambda^\theta = \operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 - z^2 - \lambda) \simeq \mathbb{C}^*$ if $\lambda \neq 0$.
Note that $\mathcal{X}_\lambda$ is smooth, so π_λ is iso, we get $\pi^{-1}(\mathcal{X}_\lambda^\theta) \simeq \mathcal{X}_\lambda^\theta \simeq \mathbb{C}^*$.

An Example



$\mathcal{X}^\theta_\lambda = (x^2 - z^2 - \lambda)$
$\mathbb{C}^*$
$\mathbb{C}^*$
λ
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$
$\pi^{-1}(\mathcal{X}^\theta_\lambda)$
$\mathbb{C}^*$
$\mathbb{A}^1$
$\mathbb{A}^1$
$\mathbb{C}^*$
$\mathbb{P}^1$
λ
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$

An Example

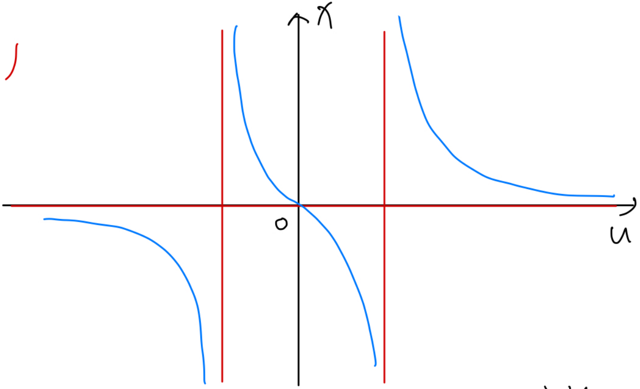

$\mathcal{X}^\theta_\lambda = (x^2 - z^2 - \lambda)$
$\mathbb{C}^*$
$\mathbb{C}^*$
λ
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$
$\pi^{-1}(\mathcal{X}^\theta_\lambda)$
$\mathbb{C}^*$
'A'
'A'
zoom in
$\mathbb{C}^*$
'P'
λ
$\lambda \neq 0$
$\lambda = 0$
$\lambda \neq 0$

An Example

$\pi^{-1}(x_\lambda^\theta)$

$\lambda = 0$

$\lambda \neq 0$



$$x(u^2-1) = 2\lambda u \Rightarrow x = \frac{2\lambda u}{u^2-1}$$

$$z = xu - \lambda$$

Kleinian singularities

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. The algebra of invariant functions $\mathbb{C}[u, v]^\Gamma$ is finitely generated.

Kleinian singularities are the quotients $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u, v]^\Gamma$.

Example

When $\Gamma = \{\pm I_2\}$, we have $\mathbb{C}[u, v]^\Gamma =$ even degree polynomials $=$ $\mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x, y, z]/(xy - z^2)$.

Fact (Klein, 1884)

$\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation), and $\mathbb{C}^2/\Gamma$ has an isolated singularity at 0.

Kleinian singularities

Classification of finite subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$
Deformations of Kleinian singularities

(A_n) cyclic group of order $(n+1)$ $\rightsquigarrow$ $xy - z^{n+1} = 0$
$xy - z^{n+1} - \sum_{i=0}^{n-1} \lambda_i z^i = 0$

(D_n) binary dihedral group of order $4(n-2)$ $\rightsquigarrow$ $x^{n-1} + xy^2 + z^2 = 0$
$x^{n-1} + \sum_{i=0}^{n-2} \lambda_i x^i + \lambda_n y + xy^2 + z^2 = 0$

(E_6) binary tetrahedral group of order 24 $\rightsquigarrow$ $x^4 + y^3 + z^2 = 0$.
$x^4 + y^3 + z^2 +$ lower order terms...

(E_7) binary octahedral group of order 48 $\rightsquigarrow$ $x^3 y + y^3 + z^2 = 0$
$x^3 y + y^3 + z^2 +$ lower order terms...

(E_8) binary icosahedral group of order 120 $\rightsquigarrow$ $x^3 y + y^3 + z^2 = 0$
$x^5 + y^3 + z^2 +$ lower order terms...

McKay correspondence: Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are in bijection with ADE Dynkin diagrams.

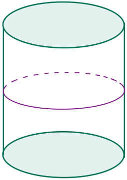
Minimal resolutions

McKay correspondence: Kleinian singularities are in bijection with ADE Dynkin diagrams.

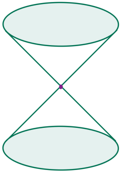
$\pi\colon Y \to X$, minimal resolution.
$\pi^{-1}(0) =$ union of $\mathbb{P}^1$'s, according dually to ADE Dynkin diagram.

Examples:

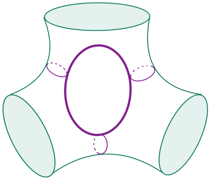
$T^*\mathbb{P}^1$



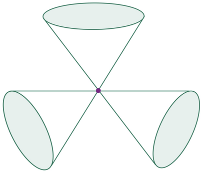
$\xrightarrow{\pi}$



A_1 singularity
$xy - z^2 = 0$



$\xrightarrow{\pi}$



D_4 singularity
$x^3 + xy^2 + z^2 = 0$

Deformations of Kleinian singularities

What are the singularities of $\mathcal{X}_\lambda,\ \lambda \in \mathfrak{h}/W$?
What is the exceptional fiber of $\pi\colon \mathcal{Y}_{\tilde\lambda} \to \mathcal{X}_\lambda,\ \tilde\lambda \in \mathfrak{h}$?

Proposition (Slodowy)

Let $p \in \mathcal{X}_\lambda$ be a singular point. Let $\Delta(\tilde\lambda) = \Delta_1 \cup \cdots \cup \Delta_m$ be the decomposition of the Dynkin diagram of the reductive group $Z_G(\tilde\lambda)$ into connected components. Then the completion of $\mathcal{X}_\lambda$ at p is a Kleinian singularity of type Δ_i for a suitable $i \in \{1, \cdots, m\}$. The exceptional fiber $\pi^{-1}(p)$ is a union of $\mathbb{P}^1$'s according dually to Δ_i.

Example: The surface $(xy - (z-1)^2(z+2) = 0)$ is a deformation of type A_2 singularity $(xy - z^3 = 0)$.
The completion of $(xy - (z-1)^2(z+2) = 0)$ at its unique singular point $p = (0,0,1)$ is a type A_1 Kleinian singularity.
Key observation: $z+2$ is invertible around $p = (0,0,1)$. We can take its square root, thus get $\frac{x}{\sqrt{z+2}}\frac{y}{\sqrt{z+2}} - (z-1)^2 = 0$.

Poisson structures

Set $X = \mathbb{C}^2/\Gamma$. The algebra of functions $\mathbb{C}[X] = \mathbb{C}[u,v]^\Gamma$ is a graded (*by degree of polynomials in* u, v) Poisson algebra with Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

Example: Type A_n Kleinian singularity $\mathbb{C}[x,y,z]/(xy - z^{n+1})$. The Poisson brackets (up to rescaling) are given by

$$\{x,y\} = (n+1)z^n,\ \{x,z\} = x,\ \{y,z\} = -y.$$

On the deformation $\mathbb{C}[x,y,z]/(xy - P(z))$, the Poisson brackets are given by

$$\{x,y\} = P'(z),\ \{x,z\} = x,\ \{y,z\} = -y.$$

Anti-Poisson involutions

Definition. An *anti-Poisson involution* of a Kleinian singularity $X = \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\}, \ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

Proposition (H., 2025)

There are finitely many θ on $\mathbb{C}^2/\Gamma$ up to conjugation. They can all be written out explicitly.

Definition. We say that θ acts on $\mathcal{X}_\lambda$ if there exists a filtered involution $\theta_\lambda\colon \mathbb{C}[\mathcal{X}_\lambda] \to \mathbb{C}[\mathcal{X}_\lambda]$ s.t. $\operatorname{gr}\theta_\lambda = \theta$. We may simply write θ_λ by θ if no confusion is caused.

Proposition (H. in progress)

Fix X and θ. Then θ acts on the deformation space $\mathfrak{h}/W$. Moreover, θ acts on $\mathcal{X}_\lambda$ if and only if $\lambda \in (\mathfrak{h}/W)^\theta$.

Fixed Point Locus

Example. Consider type A_n Kleinian singularity $xy - z^{n+1} = 0$.
(1) The anti-Poisson involution $\theta\colon x \leftrightarrow y$ acts on any deformation $xy - (z-\lambda_0)(z-\lambda_1)\cdots(z-\lambda_n) = 0$.
(2) The anti-Poisson involution $\theta\colon z \mapsto -z$ acts on the deformation $xy - (z-\lambda_0)(z-\lambda_1)\cdots(z-\lambda_n) = 0$ if and only if $\lambda_i = -\lambda_{n-i}$.

Definition. Assume θ acts on $\mathcal{X}_\lambda$. The *fixed point locus* is $\mathcal{X}_\lambda^\theta := \operatorname{Spec}\mathbb{C}[\mathcal{X}_\lambda]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[\mathcal{X}_\lambda])$.

Example: Type A_n singularity $X = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy - z^{n+1})$ with $\theta\colon x \leftrightarrow y$. The fixed-point locus
$X^\theta = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy - z^{n+1}, x-y) \simeq \operatorname{Spec}\mathbb{C}[x,z]/(x^2 - z^{n+1})$, which is a union of two $\mathbb{A}^1$'s when n is odd, a cusp when n is even.

$\mathcal{X}_\lambda^\theta$ can be computed similarly, but description is more complicated. In general, irreducible component is non-smooth affine curves.
E.g. The fixed point locus of $xy - (z-1)^2(z+2) = 0$ under $\theta\colon x \leftrightarrow y$ is $x^2 - (z-1)^2(z+2) = 0$, a nodal cubic.

Lift anti-Poisson involutions

Fix θ. Recall $\pi : \mathcal{Y}_{\tilde{\lambda}} \to \mathcal{X}_\lambda$ minimal resolution. Want to study $\pi^{-1}(\mathcal{X}_\lambda^\theta)$.

Theorem (H., 2025 – All types ADE)

For $\mathcal{X}_0 = X$ and $\pi : Y \to X$, there exists a unique anti-symplectic involution $\tilde{\theta} : Y \to Y$ such that. $\pi \circ \tilde{\theta} = \theta \circ \pi$.

Idea of Proof: quiver varieties and involutions.

Conjecture (Theorem in type A – H., in progress)

Fix $\lambda \in \mathfrak{h}/W$. For suitable choice of $\tilde{\lambda} \in \mathfrak{h}$, there exists a unique anti-symplectic involution $\tilde{\theta} : \mathcal{Y}_{\tilde{\lambda}} \to \mathcal{Y}_{\tilde{\lambda}}$ such that $\pi \circ \tilde{\theta} = \theta \circ \pi$.

Main difficulty: There are multiple choice of $\tilde{\lambda}$. Though θ fixes λ, it may permute different $\tilde{\lambda}'s$. (In the Theorem, $\lambda = 0 \Rightarrow \tilde{\lambda} = 0$) Currently working towards type D.

Preimages

Why the lift is useful?

Fact: $\mathcal{Y}_{\tilde{\lambda}} \Rightarrow \mathcal{Y}_{\tilde{\lambda}}^{\tilde{\theta}}$ smooth $\Rightarrow$ No intersection, no cusp/nodal.
We have $\pi^{-1}(\mathcal{X}_\lambda^\theta) = \mathcal{Y}_{\tilde{\lambda}}^{\tilde{\theta}} \bigcup \cup_i \pi^{-1}(p_i)$, where p_i's are the singular points of $\mathcal{X}_\lambda^\theta$ (The singular points of $\mathcal{X}_\lambda$ that are fixed by θ).

$L \subset \mathcal{X}_\lambda^\theta$ be an irreducible component. Then its normalization $\tilde{L} \subset \mathcal{Y}_{\tilde{\lambda}}^{\tilde{\theta}}$.
Irreducible components of $\pi^{-1}(\mathcal{X}_\lambda^\theta)$ are projective lines and smooth affine curves.

How to determine the preimage $\pi^{-1}(\mathcal{X}_\lambda^\theta)$?

- Draw the exceptional fibers $\pi^{-1}(p_i)$: union of $\mathbb{P}^1$'s.
- Analyze the action of $\tilde{\theta}$ on each $\pi^{-1}(p_i)$.
- Attach the smooth affine curve $\tilde{L}$ to isolated points in $\pi^{-1}(p_i)^{\tilde{\theta}}$.

Examples

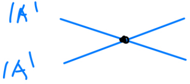
Deformations of Type A_n Kleinian singularities, with $\theta\colon x \leftrightarrow y$.
$\mathcal{X}_\lambda = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy - \prod_{i=0}^{n}(z-\lambda_i))$, $\sum_i \lambda_i = 0$.

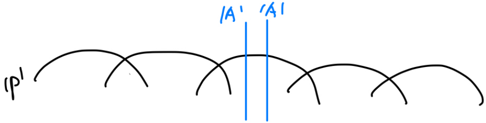
1. $\lambda = 0$

$$X = \operatorname{Spec}\mathbb{C}[x,y,z]/(xy - z^{n+1}),\ \text{type } A_n,\ \theta: x \leftrightarrow y$$

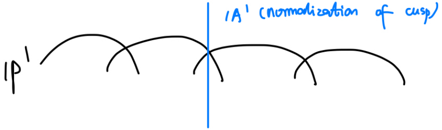
X^θ $\qquad\qquad$ $\pi^{-1}(X^\theta)$

n odd





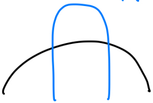
n even



2. $X_\lambda = \mathrm{Spec}\,\mathbb{C}[x, y, z] / (xy - (z-1)^2(z+2))$ $\qquad \theta : x \leftrightarrow y$

$X_\lambda^\theta = (x^2 - (z-1)^2(z+2))$ $\qquad \pi^{-1}(X_\lambda^\theta)$

nodal cubic



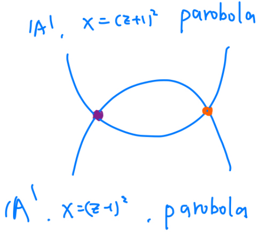
$\mathbb{P}^1$

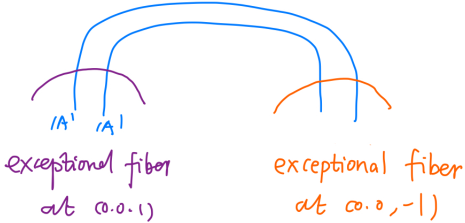
$\mathbb{A}^1$ (normalization of nodal cubic)

Examples

3. $\mathcal{X}_\lambda = \operatorname{Spec} \mathbb{C}[x, y, z] / (xy - (z-1)^2(z+1)^2) \qquad \theta : x \leftrightarrow y$

$\mathcal{X}_\lambda^\theta = (x^2 - (z-1)^2(z+1)^2) \qquad \pi^{-1}(\mathcal{X}_\lambda^\theta)$

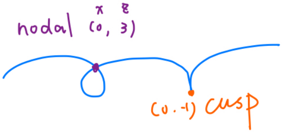


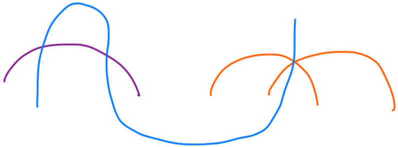


Examples

4. $X_\lambda = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy-(z-3)^2(z+1)^3(z+3))$, $\theta : x \leftrightarrow y$

$X_\lambda^\theta = (x^2-(z-3)^2(z+1)^3(z+3))$ $\qquad \pi^{-1}(X_\lambda^\theta)$





normalization of X_λ^θ : $\mathbb{C}^*$ (hyperbola)

$$\mathbb{C}[t, z]/(t^2-(z+1)(z+3)) \xrightarrow{\text{birational}} \mathbb{C}[x, z]/(x^2-(z-3)^2(z+1)^3(z+3))$$

$$(t, z) \longmapsto (x = t(z-3)(z+1), z)$$

$$\left(t = \frac{x}{(z-3)(z+1)}, z\right) \longleftarrow (x, z)$$

Thank you!